Last Time

The Green's func. for

$\Box h(\vec{r},t) = -hs(\vec{r},t)$

is

$$G = \frac{k\,\delta\left(t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right)}{4\pi|\vec{r}-\vec{r}'|}$$

& then the integral soltn. to

$\Box V = -\rho/\epsilon_0$ was

$$V(\vec{r},t) = \int_{\text{all space}} \frac{\rho(\vec{r}', t - \mathscr{r}/c)}{4\pi\epsilon_0 \mathscr{r}}\, d\tau'$$

w/ $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$

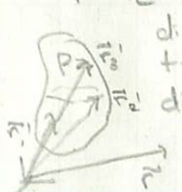
& for

$\Box \vec{A}(\vec{r},t) = -\mu_0 \vec{J}(\vec{r},t)$:

$$\vec{A} = \int_{\text{all space}} \frac{\mu_0 \vec{J}(\vec{r}', t - \mathscr{r}/c)}{4\pi \mathscr{r}}\, d\tau'$$

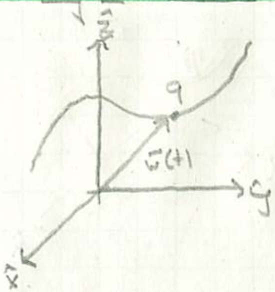
the issue w/ evaluating the expressions, beyond the integrals, is the "retarded time" calc.

$t - \mathscr{r}/c$



different $t - \mathscr{r}/c$ for different $\vec{r}'$!

Single Particle Source



a charge q moves along the trajectory $\vec{w}(t)$

$$\rho = q\,\delta^3(\vec{r} - \vec{w}(t))$$

$$\vec{J} = \rho\vec{v} = q\,\delta^3(\vec{r} - \vec{w}(t))\,\dot{\vec{w}}(t)$$

What is the potential $V(\vec{r},t)$ at location $\vec{r}$ at time t?

$$V(\vec{r},t) = \int_{\text{all space}}\int_{-\infty}^{+\infty} \frac{\rho(\vec{r}',t')}{4\pi\epsilon_0}\,\frac{\delta\left(t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right)}{|\vec{r}-\vec{r}'|}\, dt'\, d\tau'$$

$$= \frac{q}{4\pi\epsilon_0}\int_{\text{all space}}\int_{-\infty}^{+\infty} \frac{\delta^3(\vec{r}' - \vec{w}(t'))}{|\vec{r}-\vec{r}'|}\,\delta\left(t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right) dt'\, d\tau'$$

this time, use the spatial $\delta^3(\vec{r}' - \vec{w}(t'))$ to perform the spatial integration: $\vec{r}' \to \vec{w}(t')$

$$= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} \frac{\delta\left(t' - \left(t - \frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)}{|\vec{r}-\vec{w}(t')|}\, dt' \quad (*)$$

let $h(t') = t' - \left(t - \frac{|\vec{r}-\vec{w}(t')|}{c}\right)$

you will show that

$$\int_{-\infty}^{+\infty} \delta(h(t'))\,p(t')\,dt' = \left|\frac{dh(t')}{dt'}\right|^{-1} p(t')\Big|_{h(t')=0}$$

In (*), $p(t') = \frac{1}{|\vec{r}-\vec{w}(t')|}$

and we need to calculate:

$$\frac{dh(t')}{dt'} = 1 - \frac{(\vec{r}-\vec{w}(t'))\cdot\frac{d\vec{w}(t')}{dt'}}{c|\vec{r}-\vec{w}(t')|}$$

let $\vec{v}(t') = \frac{d\vec{w}(t')}{dt'}$

$$= \frac{c|\vec{r}-\vec{w}(t')| - (\vec{r}-\vec{w}(t'))\cdot\vec{v}(t')}{c|\vec{r}-\vec{w}(t')|}$$

the numerator is clearly larger than zero, no need for absolute values, &

$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{\left[c|\vec{r}-\vec{w}(t')| - (\vec{r}-\vec{w}(t'))\cdot\vec{v}(t')\right]}\Bigg|_{h(t')=0}$$

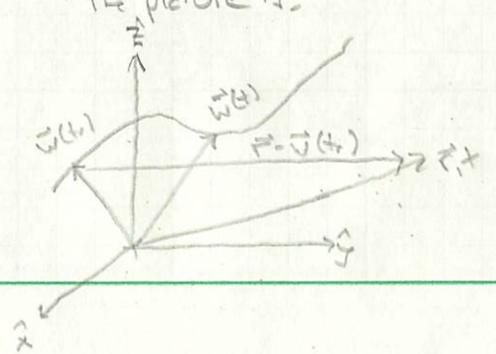
at what time t' does $h(t') = 0$?

$$t' = \left(t - \frac{|\vec{r}-\vec{w}(t')|}{c}\right) \Rightarrow c(t - t') = |\vec{r}-\vec{w}(t')|$$

& this t' is the "retarded time" often denoted t_r:

$$\underbrace{c(t - t_r)}_{\substack{\text{distance travelled}\\ \text{by light in time}\\ t - t_r}} = \underbrace{|\vec{r} - \vec{w}(t_r)|}_{\substack{\text{distance between signal "emission"}\\ \text{at } \vec{w}(t_r) \text{ \& reception at } \vec{r},t}}$$

the picture is:



let $\vec{\mathscr{r}} \equiv \vec{r} - \vec{w}(t_r)$, then we can write the potential as

$$V(\vec r,t) = \frac{q}{4\pi\epsilon_0}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec v/c)} \quad \text{w/ } \vec v \equiv \dot{\vec w}(t_r)$$

&

$$c(t-t_r) = |\vec r - \vec w(t_r)|$$ ← this constraint, while clear geometrically, is hard to evaluate

There is a new term ~ $\vec{\mathscr{r}}\cdot\vec v/c$ that would have been hard to predict — in the $v/c \ll 1$ limit, we get

$$V(\vec r,t) = \frac{q}{4\pi\epsilon_0 \mathscr{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec r - \vec w(t_r)|}$$

includes delay, but not $\mathscr{r}\cdot\vec v/c$

Magnetic Vector Potential

For $\Box \vec A = -\mu_0 \vec J$, w/ $\vec J = \rho\vec v = \rho\dot{\vec w}$, we have

$$\vec A(\vec r,t) = \frac{\mu_0 q c\,\dot{\vec w}(t_r)}{4\pi}\frac{1}{[c|\vec r - \vec w(t_r)| - (\vec r - \vec w(t_r))\cdot\dot{\vec w}(t_r)]}$$

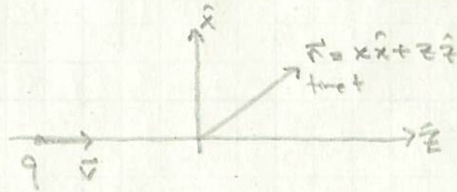
and using the same notation as above:

$$\vec A(\vec r,t) = \frac{\mu_0 q \vec v}{4\pi}\left(\frac{1}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec v/c}\right)$$

it is natural to calculate $\vec E$ + $\vec B$ from here — but the computation is complicated b/c $t_r = t - \mathscr{r}/c$ depends on both t & $\vec r$, so

$$\vec E = -\nabla V - \frac{\partial \vec A}{\partial t}$$ has terms like ∇t_r & $\frac{\partial t_r}{\partial t}$ ...

Potentials for Constant Velocity Motion



$\vec w(t) = vt\,\hat z$

$\dot{\vec w}(t) = v\,\hat z$

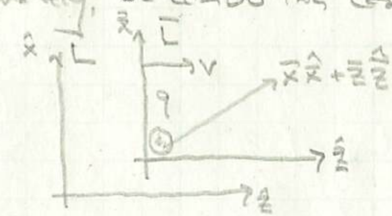
$$\vec{\mathscr{r}} = \vec r - \vec w(t_r) = x\hat x + (z - vt_r)\hat z$$

first, we need to find t_r by solving: $c(t-t_r) = \mathscr{r}$ — squaring

$$c^2(t-t_r)^2 = x^2 + (z - vt_r)^2$$

ugh — expand it, collect powers of t_r, solve the quadratic, pick the right root, ... ☹

Fortunately, we can do this case using special relativity:



In $\bar L$: $\bar V = \frac{q}{4\pi\epsilon_0}\frac{1}{(\bar x^2 + \bar z^2)^{1/2}}$

& $\bar A_z = 0$

& $A^\mu = \begin{pmatrix} V/c \\ \vec A \end{pmatrix}$ is a 4-vector, so: $V/c = \gamma(\bar V/c + v/c\,\bar A_z) = \gamma\,\bar V/c$

and $A^z = \gamma(\bar{A}^{\bar{z}} + v/c^2 \bar{V}) = \gamma v/c^2 \bar{V}$

the potential is:

$$V = \gamma \bar{V} = \frac{q\gamma}{4\pi\epsilon_0} \frac{1}{(\bar{x}^2 + \bar{z}^2)^{1/2}}$$

but we also need to express $\bar{x}$ & $\bar{z}$ in terms of the coordinates in L:

$$\bar{x} = x \qquad \bar{z} = \gamma(z - vt)$$

so $$V = \frac{q\gamma}{4\pi\epsilon_0} \frac{1}{(x^2 + \gamma^2(z-vt)^2)^{1/2}} = \frac{q}{4\pi\epsilon_0} \frac{1}{(x^2(1-v^2/c^2) + (z-vt)^2)^{1/2}}$$

& sim. $$A^z = \frac{\mu_0 q v}{4\pi} \frac{1}{(x^2(1-v^2/c^2) + (z-vt)^2)^{1/2}}$$