Poynting's Theorem

We are used to the idea that $\vec E$ & $\vec B$ carry energy along w/ them.

For a region of space w/ $\vec E$ & $\vec B$ flowing, we have

$$\frac{d}{dt}\int_\Omega u\,d\tau = -\oint_{\partial\Omega} \vec S\cdot d\vec a$$

all of the energy change in Ω is accounted for by energy flow through ∂Ω (no energy created or destroyed).

But what if there are charges in Ω, ones generating & responding to fields? In that case, the EM forces do work on the charges, & that changes the energy accounting.

Force on q due to $\vec E$ & $\vec B$ is:

$$\vec F = q\vec E + q\vec v\times\vec B$$

the work done on q by these forces is:

$$dW = \vec F\cdot d\vec l \quad \text{w/ } d\vec l = \vec v\,dt$$
$$= \vec F\cdot\vec v\,dt$$

Now $\vec F\cdot\vec v = q\vec E\cdot\vec v + \cancel{q(\vec v\times\vec B)\cdot\vec v}$
and

$$\frac{dW}{dt} = q\vec v\cdot\vec E$$

In a continuum description of charge, $q = \rho\,d\tau$, so

$$\frac{dW}{dt} = \int_\Omega \rho\vec v\cdot\vec E\,d\tau \quad \& \quad \vec J = \rho\vec v, \quad \frac{dW}{dt} = \int_\Omega \vec J\cdot\vec E\,d\tau \quad (*)$$

adding up all the contributions in Ω

We'd like a description of the work that is written entirely in terms of the fields $\vec E$ & $\vec B$ - since the charges in Ω are the source of $\vec E$ & $\vec B$, we can use Maxwell's eqn.s

$$\nabla\times\vec B = \mu_0\vec J + \mu_0\epsilon_0\frac{\partial\vec E}{\partial t} \Rightarrow \vec J = \frac{1}{\mu_0}\nabla\times\vec B - \epsilon_0\frac{\partial\vec E}{\partial t}.$$

& (*) becomes

$$\frac{dW}{dt} = \int_\Omega\left[\underbrace{\frac{1}{\mu_0}(\nabla\times\vec B)\cdot\vec E}_{\text{can be related to } \nabla\cdot(\vec E\times\vec B)} - \epsilon_0\underbrace{\vec E\cdot\frac{d\vec E}{dt}}_{=\frac{\partial}{\partial t}(\frac12 E^2)}\right]d\tau$$

$$\nabla\cdot(\vec E\times\vec B) = \vec B\cdot(\nabla\times\vec E) - \vec E\cdot(\nabla\times\vec B)$$

$$= \int_\Omega\left[\frac{1}{\mu_0}\Big(\vec B\cdot\underbrace{(\nabla\times\vec E)}_{=-\frac{\partial\vec B}{\partial t}} - \nabla\cdot(\vec E\times\vec B)\Big) - \frac{\partial}{\partial t}\left(\tfrac12\epsilon_0 E^2\right)\right]d\tau$$

$$\& \quad \vec B\cdot\frac{\partial\vec B}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac12 B^2\right)$$

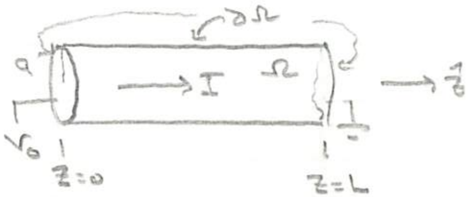
$$= -\frac{d}{dt}\int_\Omega \frac12\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \oint_{\partial\Omega}\frac{1}{\mu_0}(\vec E\times\vec B)\cdot d\vec a$$

or $$\frac{dW}{dt} = -\frac{d}{dt}\int_\Omega u\,d\tau - \oint_{\partial\Omega}\vec S\cdot d\vec a. \quad \text{work-energy theorem}$$

Poynting

In a region of space w/ no charge, $\frac{dU}{dt} = 0$ so we recover "the usual."

Example



current flows in a wire, generating magnetic field $\vec{B}$ – at the surface of the wire, $s=a$, so

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\phi}$$

the electric field set up by the potential

$$V(z) = -V_0/L\, z + V_0 \quad \text{(how do we know?)}$$

so $\vec{E} = -\nabla V = V_0/L\,\hat{z}$

Let's compute the pieces that go into Poynting's theorem:

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \tfrac{1}{2}\epsilon_0 (V_0/L)^2 + \frac{\mu_0}{2}\left(\frac{I}{2\pi a}\right)^2$$

u is t-indep., so the $\frac{d}{dt}\int_\Omega u\, d\tau$ term is zero.

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{-IV_0}{2\pi L a}\hat{s}$ at the $s=a$ surface.

since we're computing $\oint_{\partial\Omega}\vec{S}\cdot d\vec{a}$, & for the end caps, $d\vec{a} \sim \pm\hat{z} \perp \hat{s}$, we can ignore the ends

$$\frac{dW}{dt} = -\frac{d}{dt}\int_\Omega u\, d\tau - \oint \vec{S}\cdot d\vec{a} = -\int_0^L\int_0^{2\pi}\left(-\frac{IV_0}{2\pi L a}\right) a\, d\phi\, dz$$

$= IV_0$ – precisely the Joule heating from last semester.

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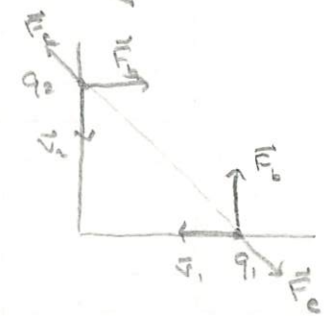
Momentum Conservation

For a charge moving w/ constant velocity:

recall the configuration from homework:



The electric forces of the charges on each other are equal & opposite.

The magnetic forces are not.

Newton's 3rd Law does not hold.

So what? Imagine a bunch of interacting masses

The total momentum of the masses in Ω is:

$$\vec{P} = \sum_{i=1}^{n} m_i \vec{v}_i$$

$+ \frac{d\vec{P}}{dt} = \sum_{i=1}^{n} m_i \dot{\vec{v}}_i = \sum_{i=1}^{n} \vec{F}_i$

Newton's third law ensures that the sum of the forces between the masses is zero, giving conservation of momentum:

$$\frac{d\vec{P}}{dt} = 0 \checkmark$$

If Newton's 3^{rd} law does not hold (& it doesn't), how do we have conservation of momentum?

particles have energy associated w/ them, & we have learned that fields do, too.

particles have momentum... is it possible that fields have momentum?

Opportunity – the Poynting vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

what are its units?

$|S| \sim \frac{\text{Power}}{\text{Area}} \sim \frac{J/s}{m^2} = \frac{kg}{s^3}$

momentum density: $\frac{kg\, m/s}{m^3} = \frac{kg}{s\, m^2}$

can you turn a kg/s^3 into a $kg/s\,m^2$ w/ some universal constant?

Sure, $\frac{kg}{s^3} = \frac{kg\, m/s}{s^2}\frac{1}{m} = \frac{kg\, m/s}{m^3}\cdot\frac{m^2}{s^2}$

just need a constant w/ units of m/s ...

$\vec{p} = \frac{\vec{S}}{c^2} = \frac{1}{\mu_0 c^2}\vec{E}\times\vec{B} = \epsilon_0 \vec{E}\times\vec{B}$

momentum density

the total momentum stored in the fields, in a volume Ω is

$$\vec{P}_\Omega = \int_\Omega \vec{p}\, d\tau.$$

For a region Ω w/ no charge in it, conservation of field momentum will take the form:

$$\frac{d}{dt}\int_\Omega \vec{p}\, d\tau = -\oint_{\partial\Omega} \_\_\_ \cdot d\vec{a}$$

object must dot into $d\vec{a}$, & leave a vector

An example of such an object: for a vector $\vec{V}$ w/ components V^i, let

$$W^{ij} = V^i V^j \doteq \begin{pmatrix} V^1V^1 & V^1V^2 & V^1V^3 \\ V^2V^1 & V^2V^2 & V^2V^3 \\ V^3V^1 & V^3V^2 & V^3V^3 \end{pmatrix} \leftarrow \text{tableaux form.}$$

Then for some other vector $\vec{u}$ w/

"$\overleftrightarrow{W}\cdot\vec{u}$" is a vector: $\sum_{j=1}^{3} W^{ij}u^j = z^i \quad i = 1,2,3$