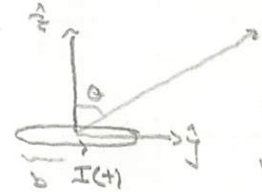
Magnetic Dipole Radiation

Starting from the integral form of the solution to

$$\Box \vec{A}(\vec{r},t) = -\mu_0 \vec{J}(\vec{r},t),$$

$$\vec{A}(\vec{r},t) = \int_{\text{all space}} \frac{\mu_0 \vec{J}(\vec{r}', t - \tfrac{\imath}{c})}{4\pi \imath} \, d\tau'$$

we specialized to $\vec{J}(\vec{r},t) = I(t)\,\delta(z)\,\delta(s-b)\,\hat{\phi}$, a circular loop of wire carrying current $I(t)$.



Taking the field point to be in the y-z plane, w/ $r \gg b$, we ended up w/

$$\vec{A}^{rad} = \frac{\mu_0 b^2 \dot{I}(t - r/c)}{4 r c} \sin\theta \, \hat{\phi}$$

The magnetic dipole moment here is

$$\vec{m} = I\vec{a} = \underbrace{I(t)\pi b^2}_{m(t)} \hat{z}$$

so

$$\vec{A}^{rad} = \frac{\mu_0 \dot{m}(t - r/c)}{4\pi r c} \sin\theta \, \hat{\phi}$$

compare w/ the electric dipole result:

" $\vec{A}_e^{rad} \cong \frac{\mu_0 \dot{p}(t - r/c)}{4\pi r} \sin\theta \, \hat{\theta}$ "

If we take $m(t) = m_0 \cos(\omega t)$ $(I(t) \sim \cos(\omega t))$ then

$$\vec{A}^{rad} = -\frac{\mu_0 m_0 \omega}{4\pi r c} \sin(\underbrace{\omega(t - r/c)}_{\equiv t_r}) \sin\theta \, \hat{\phi}$$

&

$$\vec{E}^{rad} = -\frac{\partial \vec{A}^{rad}}{dt} = \frac{\mu_0 m_0 \omega^2}{4\pi r c} \cos(\omega t_r) \sin\theta \, \hat{\phi}$$

&

$$\vec{B}^{rad} = \nabla \times \vec{A}^{rad} = -\frac{\mu_0 m_0 \omega^2}{4\pi r c^2} \cos(\omega t_r) \sin\theta \, \hat{\theta}$$

The Poynting vector is:

$$\vec{S}^{rad} = \frac{1}{\mu_0} \vec{E}^{rad} \times \vec{B}^{rad} = \frac{\mu_0}{c} \left(\frac{m_0 \omega^2}{4\pi c}\right)^2 \frac{\sin^2\theta}{r^2} \cdot \cos^2(\omega t_r) \, \hat{r}$$

w/ $\underset{\text{intensity}}{\vec{I}} = \langle \vec{S}^{rad} \rangle = \frac{\mu_0}{2c} \left(\frac{m_0 \omega^2}{4\pi c^2}\right)^2 \frac{\sin^2\theta}{r^2} \, \hat{r}$

& total power: $P_m = \oint \vec{I} \cdot d\vec{a} = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$

compare w/ $P_e = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$, the ratio is:

$$\frac{P_m}{P_e} = \frac{m_0^2}{p_0^2 c^2}$$

w/ $m_0 = \pi b^2 I_0$, $p_0 = qd$

for an oscillating charge, $I_0 \sim q\omega$

so, for comparison sake, we have:

$$\frac{P_m}{P_e} \sim \left(\frac{\pi b^2 q \omega}{q d c}\right)^2 \quad \text{taking } b \sim d,$$

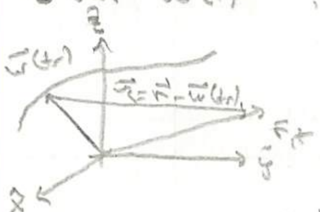
$$\sim \pi^2 \left(\frac{\omega d}{c}\right)^2$$

max speed – $\omega d/c \ll 1$.

so $P_m \ll P_e$.

Full Electric Fields

Given $\vec{w}(t)$, we can compute:



$\vec{v}(t) = \dot{\vec{w}}(t)$,
$\vec{a}(t) = \ddot{\vec{w}}(t)$,
t_r in $c(t - t_r) = \mathscr{r}$

w/ $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

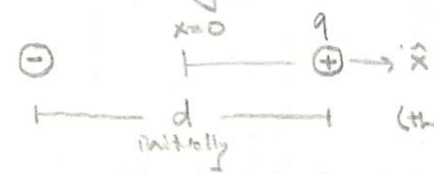
Then put the pieces together in:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right]$$

w/ all $t \to t_r$, & $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$

The piece that requires numerical evaluation is finding t_r.

Motion of Charged Particles



we'll focus on the + charge w/ position $x(t)$ (the − charge has position $-x(t)$)

The force on the + charge, due to the − charge, is:

P102: $$F = -\frac{q^2}{4\pi\epsilon_0 (x(t) - (-x(t)))^2} \quad \text{so} \quad m\ddot{x}(t) = -\frac{q^2}{4\pi\epsilon_0 \cdot 4x(t)^2}$$

& we solve for $x(t)$ w/ initial conditions: $x(0) = d/2$, $\dot{x}(0) = 0$.

If you looked at the velocity of the particle, you'd find $\dot{x}(t^*) > c$ for some finite t^*.

P201: use the relativistic form of Newton's 2nd Law:

$$\frac{d}{dt}\left[\frac{m\dot{x}(t)}{\sqrt{1 - \dot{x}(t)^2/c^2}}\right] = -\frac{q^2}{4\pi\epsilon_0 \cdot 4x(t)^2}$$

P322: The field information takes time to travel to +q: the force is:

$$F = -\frac{q^2}{4\pi\epsilon_0}\left[x(t) - (-x(t_r))\right]$$

w/ $c(t - t_r) = |x(t) - (-x(t_r))|$

but we also know that the electric field is not just the Coulomb one.

$$\frac{d}{dt}\left[\frac{m\dot{x}(t)}{\sqrt{1-\dot{x}(t)^2/c^2}}\right] = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(x(t)+x(t_r))^2}\frac{c-\dot{x}(t_r)}{c+\dot{x}(t_r)}.$$

$$c(t-t_r) = |x(t) + x(t_r)| \qquad (*)$$

+ we have initial conditions: $x(0) = d/2$
$\dot{x}(0) = 0$

But, at $t=0$, $(*)$ reads:

$$c(0-t_r) = |d/2 + x(t_r)|$$

so we need to know the location of the charges prior to $t=0$...

Suppose, for $t_0 < t < 0$, the charges are at rest at their initial locations. Then $|t_0|$ must be at least

$$-ct_0 = d \Rightarrow t_0 = -d/c$$

Whatever the charges were doing at $t < t_0$, if they are at rest at $\pm d/2$ from $t_0 \to 0$, the initial data is enough to start off.

But who guarantees the charges are at rest at $\pm d/2$ from $-d/c \to 0$?

And what about radiation reaction?