Magnetic Dipole Radiation

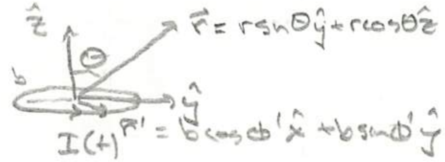
The solution to $\Box\vec{A} = -\mu_0\vec{J}$, obtained via the Green's function is:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}}\int_{-\infty}^{+\infty}\frac{\vec{J}(\vec{r}',t')}{\mathscr{r}}\,\delta(t'-(t-\mathscr{r}/c))\,dt'\,d\tau'$$

$\vec{\mathscr{r}} = \vec{r}-\vec{r}'$

$$= \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}}\frac{\vec{J}(\vec{r}',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau'$$

For the source, we'll take a circular current loop in the xy plane



$\vec{m}(t) = I(t)\,\pi b^2\,\hat{z}$

w/

$I(t) = I_0\cos(\omega t)$

$\vec{m}(t) = \underbrace{I_0\pi b^2}_{\equiv m_0}\cos(\omega t)\,\hat{z}$ is the dipole moment

& $\vec{J} = I(t)\,\delta(z)\,\delta(s-b)\,\hat{\phi}$, so

$$\vec{A} = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{I(t-\mathscr{r}/c)}{\mathscr{r}}\,\hat{\phi}\cdot b\,d\phi' \qquad (*)$$

$\hat{\phi} = -\sin\phi'\hat{x} + \cos\phi'\hat{y}$

Now, we'll assume that the "field point" $\vec{r}$

$r \gg b \Rightarrow \epsilon = b/r \ll 1$, w/

$$\vec{\mathscr{r}} = \vec{r}-\vec{r}' = -b\cos\phi'\hat{x} + (r\sin\theta - b\sin\phi')\hat{y} + r\cos\theta\hat{z}$$

$$\mathscr{r}^2 = b^2 + r^2 - 2br\sin\theta\sin\phi' = r^2(1+\epsilon^2-2\epsilon\sin\theta\sin\phi')$$

then $\mathscr{r} = r(1-2\epsilon\sin\theta\sin\phi' + \epsilon^2)^{1/2} \cong r(1-\epsilon\sin\theta\sin\phi')$

The integrand in (*) has

$$\frac{I(t-\mathscr{r}/c)}{\mathscr{r}} \approx \frac{I(t-\frac{r}{c}(1-\epsilon\sin\theta\sin\phi'))}{r(1-\epsilon\sin\theta\sin\phi')}$$

$$\approx \frac{I(t-r/c) + \epsilon\dot{I}(t-r/c)\cdot r/c\,\sin\theta\sin\phi'}{r(1-\epsilon\sin\theta\sin\phi')}$$

$$\approx \frac{1}{r}\left[I(t-r/c)(1+\epsilon\sin\theta\sin\phi') + \epsilon\dot{I}(t-r/c)\,\tfrac{r}{c}\sin\theta\sin\phi'\right]$$

$(t_r \equiv t - r/c)$

$$= \frac{I(t_r)}{r} + \frac{I(t_r)\,b}{r^2}\sin\theta\sin\phi' + \frac{b}{rc}\dot{I}(t_r)\sin\theta\sin\phi'$$

this term will integrate to zero: $\int_0^{2\pi} I\hat{\phi}\,d\phi = 0$

this term has $1/r^2$ in it, so won't contribute to radiation

then

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi r}\frac{b^2}{c}\sin\theta\,\dot{I}(t_r)\int_0^{2\pi}\underbrace{\sin\phi'(-\sin\phi'\hat{x}+\cos\phi'\hat{y})\,d\phi'}_{\text{integrates to }\pi}$$

$$= -\frac{\mu_0 b^2}{4rc}\sin\theta\,\dot{I}(t_r)\,\hat{x}$$

from the P.O.V. of $\vec{r}$, this is the $-\hat{\phi}$ direction

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 b^2\omega}{4rc}I_0\sin(\omega t_r)\sin\theta\,\hat{\phi}$$

$$= -\frac{\mu_0 m_0\omega}{4\pi rc}\sin(\omega t_r)\sin\theta\,\hat{\phi}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = +\frac{\mu_0 m_0\omega^2}{4\pi rc}\cos(\omega t_r)\sin\theta\,\hat{\phi} \quad \&$$

$$\vec{B} = \nabla\times\vec{A} = -\frac{\mu_0 m_0\omega^2}{4\pi rc^2}\cos(\omega t_r)\sin\theta\,\hat{\theta}$$