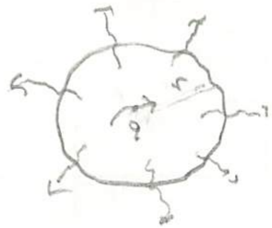
General Electric Dipole Radiation



Given a charge moving along $\vec{w}(t)$, we have:

$\vec{p} = q\vec{w}(t)$, & the radiation fields are:

$$\vec{E}^{rad} = \frac{\mu_0 q}{4\pi r}\hat{r}\times(\hat{r}\times\ddot{\vec{p}}(t-r/c))$$

$$\vec{B}^{rad} = -\frac{\mu_0 q}{4\pi r c}\hat{r}\times\ddot{\vec{p}}(t-r/c)$$

w/ $$\vec{S}^{rad} = \frac{\mu_0}{16\pi^2 r^2 c}\ddot{p}(t-r/c)^2\sin^2\Theta\,\hat{r} \qquad (+)$$

The total power radiated away is:

$$P = \oint_{\partial\Omega}\vec{S}^{rad}\cdot d\vec{a} = \frac{\mu_0\ddot{p}(t-r/c)^2}{6\pi c} \qquad (*)$$

These expressions are true for any dipole

$$\vec{p}(t) = \int_{\text{all space}} \rho\vec{r}\,d\tau$$

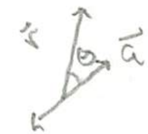
When we specialize to point charges,

$\ddot{\vec{p}} = q\ddot{\vec{w}} = q\vec{a}$, & (*) gives Larmor's Formula:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

- q^2, both + & − radiate the same amount
- a^2, speeding up & slowing down radiate the same amount.

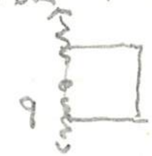
From (+), we can see the local profile:



$S^{rad} \sim \sin^2\Theta$

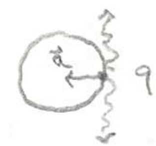
examples: 1. "stopping radiation"

? in what direction is the radiation?



2. "synchrotron radiation"

? what is the direction of greatest radiation?

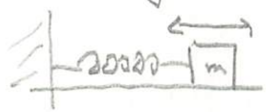


3. Hydrogen:

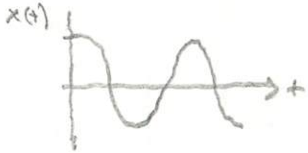
power is lost to radiation, cause e^- to fall in to the nucleus - how long does that take?

Radiation Reaction

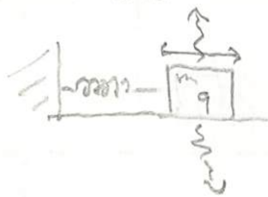
An uncharged mass on a spring:

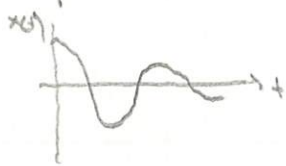


oscillates back & forth "forever."



If the mass has charge q:

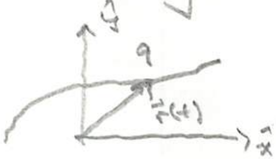




losing energy "damps" the motion. We can view this damping as an effective force

Idea - come up w/ an expression for $\vec{F}$ that leads to the same power loss as radiation.

For a charge moving a path, $\vec{r}(t)$:



in a time dt, the particle moves:

$$d\vec{l} = \frac{d\vec{r}}{dt}dt = \vec{v}dt$$

the work done by a force $\vec{F}$ over this interval is:

$$dW = \vec{F}\cdot d\vec{l} = \vec{F}\cdot\vec{v}\,dt$$

the total work done for $t = 0 \to T$ is

$$W = \int_0^T \vec{F}\cdot\vec{v}\,dt$$

we want this W to be: $-\int_0^T P\,dt$ ← Larmor power

Now $P = \frac{\mu_0 q^2}{6\pi c}\vec{a}\cdot\vec{a}$, &

$$\int_0^T \frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}dt = \vec{v}\cdot\frac{d\vec{v}}{dt}\Big|_0^T - \int_0^T \frac{d^2\vec{v}}{dt^2}\cdot\vec{v}\,dt \quad \text{(int. by parts)}$$

assume $\vec{v}(t+T) = \vec{v}(t)$ (oscillatory)

$$= 0 - \int_0^T \dot{\vec{a}}\cdot\vec{v}\,dt$$

so that $-\int_0^T P\,dt = \int_0^T \frac{\mu_0 q^2}{6\pi c}\dot{\vec{a}}\cdot\vec{v}\,dt$

compare w/ $\int_0^T \vec{F}\cdot\vec{v}\,dt$ → $\boxed{\vec{F} = \frac{\mu_0 q^2 \dot{\vec{a}}}{6\pi c}}$

Example: $x(t) = d\cos(\omega t)$ has

$$\dot{x}(t) = -d\omega\sin(\omega t)$$

$$\ddot{x}(t) = -d\omega^2\cos(\omega t) = -\omega^2 x(t)$$

$$\dddot{x}(t) = d\omega^3\cos(\omega t) = -\omega^2\dot{x}(t)$$

so $F = -\frac{\mu_0 q^2\omega^2}{6\pi c}\dot{x}$ a damping force from P201.

there must be a driving force here, to sustain the motion – that force provides the energy that is radiated.

example: free particle: $\overset{m}{\underset{q}{\bullet}}$ has

$$m\ddot{x} = \frac{\mu_0 q^2 \dot{a}}{6\pi c} \Rightarrow \ddot{x} = \underbrace{\frac{\mu_0 q^2}{6\pi m c}}_{\text{const, w/ units of } 1/s\text{, call it } \tau} \dddot{x}$$

then:

$$\ddot{x}(t) = \tau \dddot{x}(t) \quad \text{has}$$

$$\ddot{x}(t) = a_0 e^{t/\tau}$$

Fine as long as $a_0 = 0$ ...

Landau-Lifschitz Force

An alternate formulation that sidesteps some of the issues.

For a charge subject to an applied force F, we have:

$$m\ddot{x}(t) = F(t) + m\tau\dddot{x}(t)$$

or

$$m(\ddot{x}(s) - \tau\dddot{x}(s)) = F(s)$$

using s as the time-parameter label.

$$m\ddot{x}(s-\tau) \approx F(s)$$

by (inverse) Taylor expansion,

let $t \equiv s - \tau$, then

$$m\ddot{x}(t) = F(t+\tau) \approx F(t) + \tau\dot{F}(t)$$

and the radiation piece is

example: $F = -kx$ has $\dot{F} = -k\dot{x}$, so

$$m\ddot{x} = -kx \underbrace{-\tau k\dot{x}}_{\text{a damping force, again.}}$$

free particle has $F=0$, so $m\ddot{x} = 0$ ✓