Larmor Formula

For our dipole model: $\updownarrow q$ $\vec{w}(t) = d\cos(\omega t)\hat{z}$ w/
we found: $\vec{p}(t) = qd\cos(\omega t)\hat{z}$

$$\vec{S}^{rad} = \frac{\mu_0}{c}\left(\frac{q\omega^2 d}{4\pi}\right)^2 \frac{\sin^2\theta}{r^2}\cos^2(\omega(t - r/c))\hat{r}$$

+ $[\omega^2 d\cos(\omega(t - r/c))]^2$ looks $\ddot{p}(t - r/c)^2$

magnitude-squared of 2nd derivative of the dipole moment.

so $$\vec{S}^{rad} = \frac{\mu_0}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\ddot{p}(t - r/c)^2\hat{r}$$

The total power leaving a sphere of radius r is:

$$P = \oint \vec{S}^{rad}\cdot d\vec{a} = \frac{\mu_0}{16\pi^2 c}\ddot{p}(t - r/c)^2 \int_0^{2\pi}\int_0^{\pi}\frac{\sin^2\theta}{r^2}r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0}{8\pi c}\ddot{p}(t - r/c)^2\cdot\frac{4}{3} = \frac{\mu_0\ddot{p}(t - r/c)^2}{6\pi c} \quad (*)$$

For a charge moving along a given $\vec{w}(t)$, we have

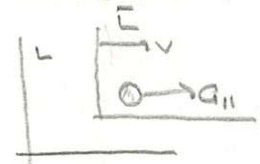
$\vec{p} = q\vec{w} \rightarrow \dot{\vec{p}} = q\vec{v} \rightarrow \ddot{\vec{p}} = q\vec{a}$, + (*) becomes:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$ the Larmor formula.

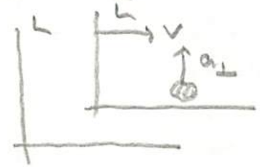
• note that a^2 appears – accelerating + ~~decelerating~~ both cause radiation.

we can find the relativistic form by putting the slowly moving charge in $\bar{L}$ (its rest frame) + moving it through L at speed v.

There are 2 cases to consider:



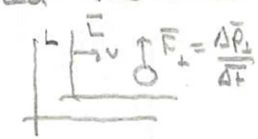
+



Acceleration Parallel to Boost

First we need the force transformations:

1. $F_{\perp}$



$$F_{\perp} = \frac{\Delta p_{\perp}}{\Delta t} = \frac{\Delta\bar{p}_{\perp}}{\gamma(\Delta\bar{t} + v/c^2\,\cancel{\Delta\bar{x}})} = \frac{1}{\gamma}\bar{F}_{\perp}$$

as before.

2. $F_{\parallel}$ $\quad \bar{F}_{\parallel} = \frac{\Delta\bar{p}_{\parallel}}{\Delta\bar{t}}$ w/ $F_{\parallel} = \frac{\Delta p_{\parallel}}{\Delta t} = \frac{\gamma(\Delta\bar{p}_{\parallel} + v/c^2\,\cancel{\Delta\bar{E}})}{\gamma(\Delta\bar{t} + v/c^2\,\cancel{\Delta\bar{x}})} = \bar{F}_{\parallel}$

Newton's 2nd law relates $F_{\parallel}$ to $a_{\parallel}$:

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1 - \dot{x}^2/c^2}}\right) = F_{\parallel} \Rightarrow \frac{m\ddot{x}(1 - \dot{x}^2/c^2) + m\dot{x}\,\ddot{x}/c^2}{(1 - \dot{x}^2/c^2)^{3/2}} = F_{\parallel}$$

giving $$m a_{\parallel} = F_{\parallel}(1 - v^2/c^2)^{3/2} \quad (+)$$

now for power – in $\bar{L}$: using $\bar{a}_{\parallel} = \bar{F}_{\parallel}/m$ (no $\bar{v}$)

using (+)

$$\bar{P} = \frac{\mu_0 q^2\bar{a}_{\parallel}^2}{6\pi c} = \frac{\mu_0 q^2\bar{F}_{\parallel}^2}{6\pi m^2 c} = \frac{\mu_0 q^2 F_{\parallel}^2}{6\pi m^2 c} = \frac{\mu_0 q^2 a_{\parallel}^2}{6\pi c}(1 - v^2/c^2)^{-3}$$

so $\bar{P} = P$ gives: $P = \frac{\mu_0 q^2 a_{\parallel}^2}{6\pi c}\gamma^6$

this can get large!

Acceleration Perpendicular to Boost

In general: $\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) = \vec{F}$ has

$$\frac{m\vec{a}}{\sqrt{1-v^2/c^2}} - \frac{\frac{1}{2}m\vec{v}\,(-2\,\vec{v}\cdot\vec{a}/c^2)}{(1-v^2/c^2)^{3/2}} = \vec{F}$$

& for $\vec{a} \perp$ to $\vec{v}$ (boost), $\vec{v}\cdot\vec{a}_\perp = 0$, so

$$m a_\perp = F_\perp (1-v^2/c^2)^{1/2}$$

picking one of the $\perp$ directions.

$\bar{F}_\perp = \gamma F_\perp$

Then: $\bar{P} = \frac{\mu_0 q^2 \bar{a}_\perp^2}{6\pi c} = \frac{\mu_0 q^2 \bar{F}_\perp^2}{6\pi m^2 c} = \frac{\mu_0 q^2 F_\perp^2 \gamma^2}{6\pi m^2 c} = \frac{\mu_0 q^2 a_\perp^2 \gamma^4}{6\pi c}$

& $\bar{P} = P = \frac{\mu_0 q^2 a_\perp^2 \gamma^4}{6\pi c}$

Combining the two: $P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$

Liénard's generalization of the Larmor formula.