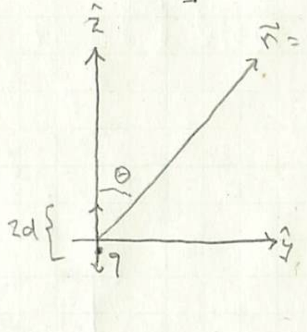
Oscillating Charge Radiation



$\vec r = r\sin\theta\,\hat y + r\cos\theta\,\hat z$

charge q moves according to: $\vec w(t) = d\cos(\omega t)\hat z$

w/ dipole moment

$$\vec p(t) = q\vec w(t) = qd\cos(\omega t)\hat z$$

We assumed that $\epsilon \equiv \frac{d}{r} \ll 1$ & $\delta \equiv \frac{\omega d}{c} \ll 1$, to get, for the (approximate) potential at $\vec r, t$:

$$V(\vec r,t) = \frac{q}{4\pi\epsilon_0 r} + \underbrace{\frac{qd\cos(\omega(t-r/c))}{4\pi\epsilon_0 r^2}\cos\theta}_{=\frac{\vec p(t-r/c)\cdot\hat r}{4\pi\epsilon_0 r^2}} - \underbrace{\frac{q\omega d\sin(\omega(t-r/c))}{4\pi\epsilon_0 rc}\cos\theta}_{=\frac{\dot{\vec p}(t-r/c)\cdot\hat r}{4\pi\epsilon_0 rc}}$$

letting $t_r \equiv t - r/c$, we can write the potential as:

$$V(\vec r,t) = \frac{q}{4\pi\epsilon_0 r} + \frac{\vec p(t_r)\cdot\hat r}{4\pi\epsilon_0 r^2} + \frac{\dot{\vec p}(t_r)\cdot\hat r}{4\pi\epsilon_0 rc}$$

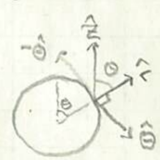
only the last term will contribute to radiation.

The magnetic vector potential is:

$$\vec A(\vec r,t) = \frac{\vec v\, V(\vec r,t)}{c^2} = -\frac{\omega d\sin(\omega t_r)\hat z}{c\cdot c}\left[\frac{q}{4\pi\epsilon_0 r}\left(1+O(\epsilon)+O(\delta)\right)\right]$$

δ''

$$= -\frac{q\omega d\sin(\omega t_r)\hat z}{4\pi\underbrace{\epsilon_0 c^2}_{''1/\mu_0} r}$$



$\hat z = \cos\theta\,\hat r - \sin\theta\,\hat\theta$

the "radiation potentials" are:

$$V^{rad} = -\frac{q\omega d\sin(\omega t_r)}{4\pi\epsilon_0 rc}\cos\theta$$

$t_r = t - r/c$

$$\vec A^{rad} = -\frac{\mu_0 q\omega d\sin(\omega t_r)}{4\pi r}\left(\cos\theta\,\hat r - \sin\theta\,\hat\theta\right)$$

let's look first at contributions to $\vec E$:

$$\nabla V^{rad} = \frac{\partial V^{rad}}{\partial r}\hat r + \overset{\sim 1/r^2 \text{ drop it}}{\frac{1}{r}\frac{\partial V^{rad}}{\partial\theta}\hat\theta}$$

$$= \left[\overset{\sim 1/r^2 \text{ drop it}}{\frac{q\omega d\sin(\omega t_r)}{4\pi\epsilon_0 r^2 c}} - \frac{q\omega^2 d\cos(\omega t_r)}{4\pi\epsilon_0 rc}\cos\theta\left(-\frac{1}{c}\right)\right]\hat r$$

$$\nabla V^{rad}\Big|_{r^{-2}\text{ terms remove}} = +\frac{\mu_0 q\omega^2 d}{4\pi r}\cos(\omega t_r)\cos\theta\,\hat r$$

we also need:

$$\frac{\partial\vec A^{rad}}{\partial t} = -\frac{\mu_0 q\omega^2 d}{4\pi r}\cos(\omega t_r)\left[\cos\theta\,\hat r - \sin\theta\,\hat\theta\right]$$

then

$$\vec E^{rad} = -\left(\nabla V^{rad} + \frac{\partial\vec A^{rad}}{\partial t}\right)\Big|_{r^{-2}\text{ terms removed}}$$

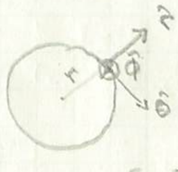
$$= -\frac{\mu_0 q\omega^2 d}{4\pi r}\cos(\omega(t-r/c))\sin\theta\,\hat\theta.$$

the $\hat r$ terms cancel.

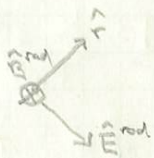
From $\vec{B}^{rad} = \frac{1}{c}\hat{z}\times\vec{E}^{rad} = \frac{1}{c}\hat{r}\times\vec{E}^{rad}$ ← here.

$$= -\frac{\mu_0 q\omega^2 d}{4\pi r c}\cos(\omega(t-r/c))\sin\Theta\,\hat{\phi}$$

At a fixed-r-radius sphere (for r large, so that our radiation fields are relevant)



we have



$$\vec{E} = \underbrace{\frac{\mu_0 q\omega^2 d}{4\pi r}\sin\Theta}_{=E_0,\ \text{a constant at fixed } r,\Theta}\cos\Big(\omega t - \underbrace{\frac{\omega r}{c}}_{\text{constant phase factor for fixed } r,\Theta}\Big)\hat{\Theta}$$

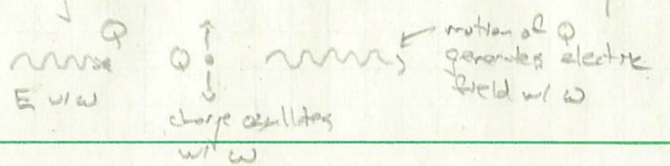
$$= E_0\cos(\omega t-\phi)\hat{\Theta} \quad \& \quad \vec{B} = \frac{E_0}{c}\cos(\omega t-\phi)\hat{\phi}$$

these are, locally, plane waves.

Note that the source charge oscillation freq. ω is also the frequency of the field.

Recall that a field like $E_0\cos(\omega t)$ makes a charge Q oscillate up & down w/ freq. ω.

But once that charge Q oscillates w/ freq. ω, it generates an electric field w/ freq. ω.



Radiated Power

The $\vec{E}^{rad}$ & $\vec{B}^{rad}$ propagate radially in all directions w/

$$\vec{S}^{rad} = \frac{1}{\mu_0}\vec{E}^{rad}\times\vec{B}^{rad} = \frac{\mu_0}{c}\left(\frac{q\omega^2 d}{4\pi r}\sin\Theta\right)^2\cos^2(\omega(t-r/c))\,\hat{r}$$

Remember the picture that defined radiation:

$$\frac{d}{dt}\int_\Omega u\,d\tau = -\oint_{\partial\Omega}\vec{S}^{rad}\cdot d\vec{a}$$

& our target was $\vec{S} = \frac{\alpha}{r^2}\hat{r}$

& we see that $\vec{S}^{rad}$ has this form $\sim 1/r^2\,\hat{r}$

units for $\vec{S}$ are power/area.

"Intensity" is the time-average of $\vec{S}^{rad}$ over one full cycle (of oscillation).

$$\vec{I} = \langle \vec{S}^{rad}\rangle = \frac{1}{T}\int_0^T \vec{S}^{rad}\,dt \quad \text{w/} \quad \omega T = 2\pi \Rightarrow T = 2\pi/\omega.$$

$$= \frac{\mu_0}{c}\left(\frac{q\omega^2 d}{4\pi r}\sin\Theta\right)^2 \underbrace{\frac{1}{T}\int_0^T\cos^2(\omega(t-r/c))\,dt}_{=1/2}$$

so $\vec{I} = \frac{\mu_0 q^2\omega^4 d^2}{32\pi^2 c}\frac{\sin^2\Theta}{r^2}\hat{r}$ still a power/area.

Note that $I \sim \sin^2\Theta$ – no power radiated in the $\Theta = 0, \pi$ directions.

large amounts of power in the $\pm\pi/2$ directions.

The total radiated power is:

$$\oint_{\partial\Omega} \vec{I}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} \frac{\mu_0 q^2\omega^4 d^2}{32\pi^2 c}\,\frac{\sin^2\Theta}{r^2}\cdot r^2\sin\Theta\, d\Theta\, d\phi$$

$$= \frac{\mu_0 q^2\omega^4 d^2}{16\pi c}\underbrace{\int_0^{\pi}\sin^3\Theta\, d\Theta}_{=4/3} = \frac{\mu_0 q^2\omega^4 d^2}{12\pi c}$$

=

We used a specific charge motion to generate the dipole moment:

$$\vec{p}(t) = q\, d\cos(\omega t)\,\hat{z}$$

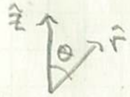
But looking at the form of $\vec{B}^{rad}$, we can generalize:

$$\vec{B}^{rad} = -\frac{\mu_0 q\,\omega^2 d}{4\pi r c}\cos(\omega(t-r/c))\sin\Theta\,\hat{\phi}$$

the ω^2 indicates 2 time-derivatives of $\cos(\omega(t-r/c))$

$\Rightarrow$

$$\ddot{\vec{p}}(t-r/c) = -q\omega^2 d\cos(\omega(t-r/c))\,\hat{z}$$

note that  has $\hat{z}\times\hat{r} = \sin\Theta\,\hat{\phi}$, appearing in $\vec{B}^{rad}$

so we can write $\vec{B}^{rad}$ as:

$$\vec{B}^{rad} = -\frac{\mu_0}{4\pi r c}\,\hat{r}\times\ddot{\vec{p}}(t-r/c)$$

and for $\vec{E}^{rad} = -\frac{\mu_0 q\,\omega^2 d}{4\pi r}\cos(\omega(t-r/c))\sin\Theta\,\hat{\Theta}$

$\hat{r}\times\hat{\phi} = \hat{\Theta}$

$$= \frac{\mu_0}{4\pi r}\,\hat{r}\times(\hat{r}\times\ddot{\vec{p}}(t-r/c)).$$