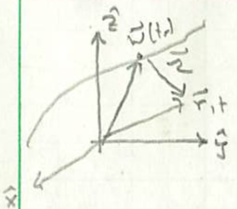
Last Time



Given $\vec{w}(t)$,
$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$,
$\vec{v} = \dot{\vec{w}}(t_r)$,
$\vec{a} = \ddot{\vec{w}}(t_r)$ w/

$\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$, then:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{u}\cdot\vec{\mathscr{r}})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})\right]$$

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}} \times \vec{E}$$

$$c(t - t_r) = \mathscr{r}.$$

We thought about the $\vec{E}$ from a charge q moving w/ constant $\vec{v}$.

Now, we turn to the new piece,

$$\frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\vec{u}\cdot\vec{\mathscr{r}})^3} \vec{\mathscr{r}} \times (\vec{u} \times \vec{a})$$

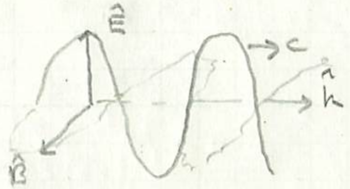
note that this goes like $1/\mathscr{r}$, not $1/\mathscr{r}^2$...

Plane Waves

These are solutions to Maxwell's eqn.s in vacuum:

$$\nabla\cdot\vec{E} = 0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}.$$



$$\vec{E} \perp \vec{B} \perp \vec{k}$$

they propagate in empty space, carrying energy (density)

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

conservation reads:

$$\frac{\partial u}{\partial t} = -\nabla\cdot\vec{S} \quad \text{w/ } \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}.$$

Whatever generated these waves must have lost energy.

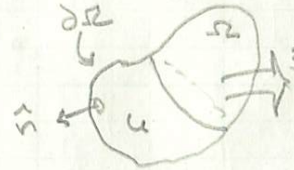
? Candidates: • q at rest

constant velocity

non-constant velocity (acceleration).

Electromagnetic Radiation

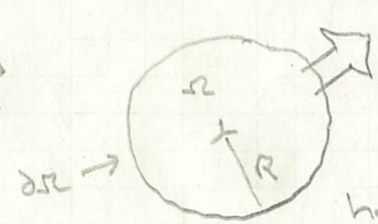
Integral form of energy conservation:



$$\underbrace{\frac{d}{dt}\int_\Omega u\, d\tau}_{\text{power}} = -\underbrace{\oint_{\partial\Omega} \vec{S}\cdot d\vec{a}}_{\text{power}}$$

$\vec{S} \parallel \hat{n}$ accounts for energy leaving Ω

Make Ω all space, the fields $\vec{E}$ & $\vec{B}$ that carry energy all the way out to $R \to \infty$ are "radiation fields"



The associated $\vec{S} = \alpha \frac{\hat{r}}{r^2}$ (α const.)

has: $\oint_{\partial\Omega} \vec{S}\cdot d\vec{a} = 4\pi\alpha$

same $\forall$ values of R including $R \to \infty$.

To get $\vec{S} = \alpha\frac{\hat{r}}{r^2} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$, we must have both

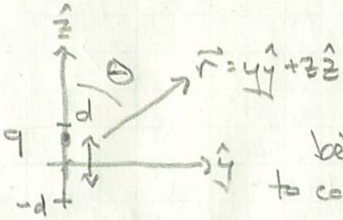
$$E \sim 1/r \quad \& \quad B \sim 1/r$$

& we already saw that the piece of $\vec{E}$ that goes like $1/r$ comes from charge acceleration

We'll take the simplest model, & squeeze out as much prediction as possible before actually calculating anything.

Oscillating Dipole Potential (Physics 321)

$$\vec{w}(t) = d\cos(\omega t)\,\hat{z}$$



we're interested in the behavior of $\vec{E}$ & $\vec{B}$ as $r \to \infty$. to capture the $1/r$ pieces

so assume that $d/r \ll 1$ & let $\epsilon \equiv d/r$.

The dominant term in V is the monopole

$$V(\vec{r},t) \cong \frac{q}{4\pi\epsilon_0 r}$$ then the dipole piece

$$V(\vec{r},t) \cong \frac{q}{4\pi\epsilon_0 r} + \frac{\vec{p}(t)\cdot\hat{r}}{4\pi\epsilon_0 r^2} \quad \text{w/} \quad \vec{p}(t) = q\vec{w}(t)$$

or, if we're feeling 322-ey, maybe $t \to t_r \cong t - r/c$

$$V(\vec{r},t) \cong \frac{q}{4\pi\epsilon_0 r} + \frac{\vec{p}(t - r/c)\cdot\hat{r}}{4\pi\epsilon_0 r^2}.$$

but none of these terms will lead to an $\vec{E} = -\nabla V$ that has $1/r$ dependence, there must be something new in the Liénard-Wiechert potentials that we cannot simply guess or identify from our previous work

Start w/ $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}/c}\ \ldots$

Oscillating Dipole Potential (Physics 322)

Same setup as on the left, but now,

$$\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r) = y\hat{y} + (z - d\cos(\omega t_r))\hat{z},$$

take $y = r\sin\theta$, $z = r\cos\theta$ to get r in place (for eventual $d/r \ll 1$ use)

$$\mathscr{r} = \left[r^2 - 2dr\cos\theta\cos(\omega t_r) + d^2\cos^2(\omega t_r)\right]^{1/2}$$

$$= r\left[1 - 2\epsilon\cos\theta\cos(\omega t_r) + \epsilon^2\cos^2(\omega t_r)\right]^{1/2}$$

$$\approx r(1 - \epsilon\cos\theta\cos(\omega t_r))$$

The retarded time condition:

$$c(t - t_r) = \mathscr{r} \approx r + O(\epsilon)$$

giving: $\quad t_r \approx t - r/c + O(\epsilon)$

Note that every time t_r appears, it is in a function w/ an ϵ out front, so we only need to keep the coarsest, $O(\epsilon^0)$ term:

$$\epsilon f(t_r) = \epsilon f(t - r/c + O(\epsilon))$$

$$\approx \epsilon\left[f(t - r/c) + \dot{f}(t - r/c)\cdot O(\epsilon)\right]$$

$$= \epsilon f(t - r/c) + O(\epsilon^2)$$

so take $t_r \approx t - r/c$.

We have $\mathscr{r} \approx r[1-\epsilon\cos\theta\cos(\omega t_r)]$ $(t_r \approx t - r/c)$

We need $\frac{\vec{\mathscr{r}}\cdot\vec{v}}{c}$ w/ $\vec{v} = \dot{\vec{w}}(t_r) = -\omega d\sin(\omega t_r)\hat{z}$

so that

$$\frac{\vec{\mathscr{r}}\cdot\vec{v}}{c} = -r(\cos\theta - \epsilon\cos(\omega t_r))\underbrace{\frac{\omega d}{c}}_{\equiv\delta \ll 1 \text{ for } \omega d \ll c}\sin(\omega t_r)$$

we assume the charge oscillates w/ maximum speed much much less than c:

$$\approx -r\delta\cos\theta\sin(\omega t_r) \quad + \mathcal{O}(\overbrace{\epsilon\delta}^{\text{squared in small}}) \text{ quantities}$$

The denominator in $V = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}/c}$ is

$$\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}/c \approx r\left[1 - \epsilon\cos\theta\cos(\omega t_r) + \delta\cos\theta\sin(\omega t_r)\right]$$

And:

$$V = \frac{q}{4\pi\epsilon_0}\left(\mathscr{r} - \frac{\vec{\mathscr{r}}\cdot\vec{v}}{c}\right)^{-1} \approx \frac{q}{4\pi\epsilon_0 r}\left[1 + \frac{d}{r}\cos\theta\cos(\omega t_r) - \frac{\omega d}{c}\cos\theta\sin(\omega t_r)\right]$$

$$\approx \underbrace{\frac{q}{4\pi\epsilon_0 r}}_{\text{monopole term ✓}} + \underbrace{\frac{qd\cos(\omega t_r)}{4\pi\epsilon_0 r^2}\cos\theta}_{= \frac{\vec{p}(t - r/c)\cdot\hat{r}}{4\pi\epsilon_0 r^2} \text{ dipole term ✓}} - \boxed{\frac{q\omega d}{4\pi\epsilon_0 rc}\sin(\omega t_r)\cos\theta}$$

$t_r = t - r/c$

new term!

The new piece looks like:

$$\frac{\dot{\vec{p}}(t - r/c)\cdot\hat{r}}{4\pi\epsilon_0 rc}$$