Helmholtz Equation Green's Function

$$\nabla^2 h(\vec{r}) + \underset{\text{const.}}{\mu^2} h(\vec{r}) = -k\, \underset{\text{source func.}}{s(\vec{r})}$$

Green's function problem has $s = \delta^3(\vec{r}-\vec{r}\,')$

$$\nabla^2 G(\vec{r},\vec{r}\,') + \mu^2 G(\vec{r},\vec{r}\,') = -k\,\delta^3(\vec{r}-\vec{r}\,')$$

1. set source location at $\vec{r}\,'=0$, assume spherical symmetry:
$$G(\vec{r},0) = G(r)$$

2. Solve: $\nabla^2 G(r) + \mu^2 G(r) = -k\delta^3(\vec{r})$ at points w/ $\vec{r} \neq 0$:
$$\frac{1}{r}(rG)'' + \mu^2 G = 0 \Rightarrow (rG)'' = -\mu^2 (rG)$$

let $H \equiv rG$, we have
$$H'' = -\mu^2 H \Rightarrow H = Ae^{\pm i\mu r}$$

so
$$G(r) = \frac{Ae^{\pm i\mu r}}{r}$$

3. Set A using the $\delta^3(\vec{r})$ at the origin as a "boundary condition" by integrating.

we already know the answer for $\mu = 0$, $A = \frac{k}{4\pi}$,
so
$$G(r) = \frac{ke^{\pm i\mu r}}{4\pi r}.$$

4. move the source back to $\vec{r}\,'$, then $r \longrightarrow \imath \equiv |\vec{r}-\vec{r}\,'|$
$$G(\vec{r},\vec{r}\,') = \frac{ke^{\pm i\mu \imath}}{4\pi \imath}$$

Fourier Transform

For a function $\rho(t)$ (really $\rho(\vec{r},t)$, but focus on the t dependence)

$$\tilde{\rho}(f) \equiv \int_{-\infty}^{+\infty} e^{i2\pi f t} \rho(t)\,dt \quad \text{w/ inverse:}$$

$$\rho(t) = \int_{-\infty}^{+\infty} e^{-i2\pi f t} \tilde{\rho}(f)\,df$$

note that self-consistency requires:

$$\rho(t) = \int_{-\infty}^{+\infty} e^{-i2\pi f t} \underbrace{\left[\int_{-\infty}^{+\infty} e^{i2\pi f\bar{t}} \rho(\bar{t})\,d\bar{t}\right]}_{=\tilde{\rho}(f)} df$$

$$= \int_{-\infty}^{+\infty} \rho(\bar{t}) \underbrace{\left[\int_{-\infty}^{+\infty} e^{i2\pi f(\bar{t}-t)}\,df\right]}_{=\delta(\bar{t}-t) \text{ must be, to get equality}} d\bar{t}$$

so a representation of the Dirac delta is:

$$\int_{-\infty}^{+\infty} e^{i2\pi f(\bar{t}-t)}\,df = \delta(\bar{t}-t)$$

D'Alembertian Green's Function

We want to solve 4 copies of

$$\Box h(\vec r,t) = -\mu_0 s(\vec r,t) \text{, or}$$

$$\left[-\frac{1}{c^2}\frac{\partial^2 h}{\partial t^2} + \nabla^2 h\right] = -\mu_0 s$$

the Green's function problem has $s = \delta^3(\vec r-\vec r\,')\delta(t-t')$.

Set the "source" at the origin, $\vec r\,'=0$ at $t'=0$, then $G(\vec r,t)$ solves:

$$-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} + \nabla^2 G = -\mu_0\delta^3(\vec r)\delta(t)$$

mul. by $e^{i2\pi ft}$ & integrate in time:

$$-\frac{1}{c^2}\underbrace{\int_{-\infty}^{+\infty} e^{i2\pi ft}\frac{\partial^2 G}{\partial t^2}\,dt}_{\text{use int. by parts twice}} + \nabla^2\underbrace{\int_{-\infty}^{+\infty} e^{i2\pi ft} G\,dt}_{=\tilde G(\vec r,f)} = -\mu_0\delta^3(\vec r)$$

$$-\frac{1}{c^2}\int_{-\infty}^{+\infty} G\frac{d^2}{dt^2}\left(e^{i2\pi ft}\right)dt + \nabla^2\tilde G(\vec r,f) = -\mu_0\delta^3(\vec r)$$

$$+\left(\frac{2\pi f}{c}\right)^2\tilde G + \nabla^2\tilde G = -\mu_0\delta^3(\vec r)$$

Aha – the Helmholtz Green's function eqn. w/ $\mu\to\left(\frac{2\pi f}{c}\right)^2$

$$\tilde G(\vec r,f) = \frac{\mu_0 e^{\pm i\frac{2\pi f}{c}r}}{4\pi r}$$

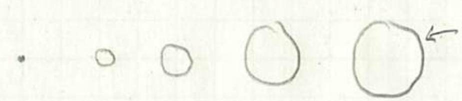
we need to use the inverse Fourier transform to get back $G(\vec r,t)$:

$$G = \int_{-\infty}^{+\infty} e^{-i2\pi ft}\,\tilde G\,df$$

$$= \frac{\mu_0}{4\pi r}\int_{-\infty}^{+\infty} e^{i2\pi f(-t\pm r/c)}\,df$$

$$= \frac{\mu_0\,\delta(t\mp r/c)}{4\pi r} \quad (\circ) \quad (\delta(-x)=\delta(x)).$$

before continuing to step 4 – think about the role of $\pm$ in G – a source flashes on & off at the origin at $t=0$ – the G tells us that an infinitesimal shell expands outward at c:



but the two solutions in $(\circ)$ go both forward & backward in time.

We want the one that goes forward in time

$$G = \frac{\mu_0\,\delta(t-r/c)}{4\pi r}$$

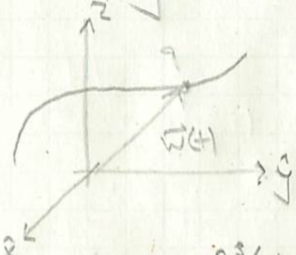
"4." $\therefore G(\vec r,\vec r\,',t,t') = \mu_0\frac{\delta(t-t'-\mathscr{r}/c)}{4\pi\mathscr{r}}$ w/ $\vec{\mathscr{r}} \equiv \vec r-\vec r\,'$

Source Issue

Our $G(\vec{r},\vec{r}',t,t')$ is for a source that "flashes" on + off instantaneously at t', location $\vec{r}'$.

Can a point charge do this? No – violates conservation of charge. ☹

Real charges move continuously through spacetime:



you can "make" this continuous trajectory out of a series of point flashes (think of holiday lights) – we can use $G(\vec{r},\vec{r}';t,t')$ w/ a source:

$$\rho = q\delta^3(\vec{r}-\vec{w}(t)) \quad + \quad \vec{J} = \rho\vec{v} = \rho\dot{\vec{w}}(t)$$

let's work on $\Box V = -\rho/\epsilon_0$

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{all}\\\text{space}}}\int_{-\infty}^{+\infty} \frac{\rho(\vec{r}',t')}{\imath}\underbrace{\delta(t-t'-\imath/c)}_{=\delta(t'-(t-\imath/c))}\,dt'\,d\tau'$$

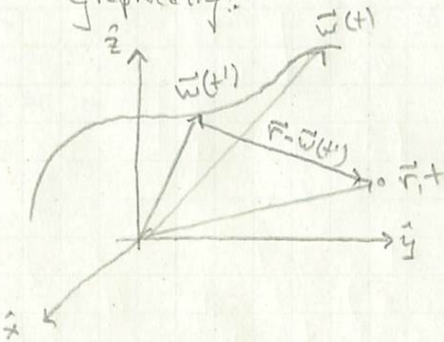
$$= \frac{q}{4\pi\epsilon_0}\int_{\substack{\text{all}\\\text{space}}}\int_{-\infty}^{+\infty} \frac{\delta^3(\vec{r}'-\vec{w}(t'))}{|\vec{r}-\vec{r}'|}\,\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{r}'|}{c}\right)\right)dt'\,d\tau'$$

$$= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} \frac{\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)}{|\vec{r}-\vec{w}(t')|}\,dt'$$

thinking of the $\delta\left(t'-\left(t-\frac{|\vec{r}-\vec{w}(t')|}{c}\right)\right)$ term, it will enforce the "retarded time" condition – it contributes when

$$t'-\left(t-\frac{|\vec{r}-\vec{w}(t')|}{c}\right) = 0 \Rightarrow \boxed{c(t-t') = |\vec{r}-\vec{w}(t')|}$$

graphically:



a "signal" travels from $\vec{w}(t')$ at time t', arriving at $\vec{r}$ at time t:

$$\underbrace{c(t-t')}_{\substack{\text{distance travelled}\\\text{by light in time}\\ t-t'}} = \underbrace{|\vec{r}-\vec{w}(t')|}_{\substack{\text{distance between}\\\text{the "field point"}\\ \vec{r} \text{ + } \vec{w}(t')}}$$

Charge density functions

If you're given a more continuous (than a point particle) $\rho(\vec{r},t)$, we have:

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{all}\\\text{space}}}\int_{-\infty}^{+\infty} \frac{\rho(\vec{r}',t')}{\imath}\,\delta(t'-(t-\imath/c))\,dt'\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{all}\\\text{space}}} \frac{\rho(\vec{r}',t-\imath/c)}{\imath}\,d\tau'$$

and similarly, given $\vec{J}(\vec{r},t)$:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{\substack{\text{all}\\\text{space}}} \frac{\vec{J}(\vec{r}',t-\imath/c)}{\imath}\,d\tau'$$