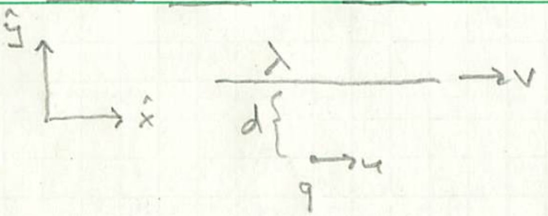
Model Problem - Review



The force on q is:

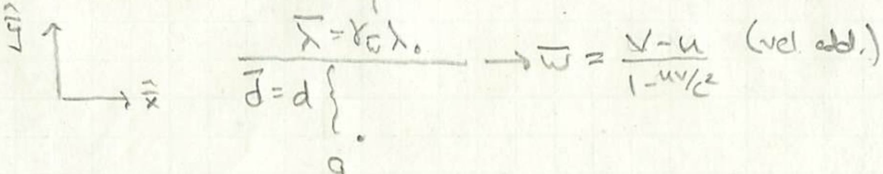
$$\vec{F} = -\frac{q\lambda}{2\pi\epsilon_0 d}\left[1-\frac{uv}{c^2}\right]\hat{y}$$

let $\gamma_p \equiv \frac{1}{\sqrt{1-p^2/c^2}}$, then $\lambda = \gamma_v \lambda_0$

w/ λ_0 the rest charge density of the line.

$$F_y = -\frac{q\lambda_0}{2\pi\epsilon_0 d}\left[1-\frac{uv}{c^2}\right]\gamma_v$$

Now - we'll analyze the same problem in the rest frame of q.



- no magnetic force here - just electric.
- we need $\bar{F}_y = \gamma_? F_y$ if SR is to hold.

which γ is this? $\quad \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}}$

In this rest frame, $\vec{E} = \frac{\bar{\lambda}}{2\pi\epsilon_0 d}(-\hat{y}) = -\frac{\gamma_{\bar{w}}\lambda_0}{2\pi\epsilon_0 d}\hat{y}$

so

$$\bar{F}_y = -\frac{q\lambda_0}{2\pi\epsilon_0 d}\cdot\gamma_{\bar{w}}$$

& from homework: $\gamma_{\bar{w}} = \gamma_u\gamma_v(1-uv/c^2)$

$$= \underbrace{-\frac{q\lambda_0}{2\pi\epsilon_0 d}\left[1-\frac{uv}{c^2}\right]\cdot\gamma_v}_{=F_y}\cdot\gamma_u$$

$$\bar{F}_y = \gamma_u F_y \checkmark$$

E&M is where special relativity was "discovered," so this result is more of a confirmation than a test.

Maxwell's Eqn.s & Special Relativity -

$\nabla\cdot\vec{E} = \rho/\epsilon_0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

hard to see how things work out - while $\vec{E}$ & $\vec{B}$ have units related by c, there are $\underline{6}$ quantities here...

$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \nabla\times\vec{A}$, V/c & $\vec{A}$ have the same units.

In Lorenz gauge, $\nabla\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$, Maxwell's eqn.s become: ✓

$$\Box V = -\rho/\epsilon_0 \qquad \Box \vec{A} = -\mu_0 \vec{J} \qquad (*)$$

w/ $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$

Now, ρ & $\vec{J}$ are also a set of 4 quantities, & $\vec{J} = \rho\vec{v}$, so ρc & $\vec{J}$ have the same units.

One can show that $A^\mu \doteq \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix}$ & $J^\mu \doteq \begin{pmatrix} \rho c \\ \vec{J} \end{pmatrix}$

are 4-vectors — under a Lorentz boost, both have:

$$\bar{A}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu \qquad \bar{J}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu J^\nu$$

the "D'Alembertian", $\Box$, is a scalar (like proper time):

$$\bar{\Box} = \Box$$

The content of $(*)$ is then:

$$\Box A^\mu = -\mu_0 J^\mu \text{ (for } \mu = 0,1,2,3)$$

As we have discussed previously, the 4-vector eqn. ensures that the "laws of physics hold in all (inertial) reference frames." ✓

In $\bar{L}$: $\bar{\Box}\bar{A}^\mu = -\mu_0 \bar{J}^\mu$

$$\Box \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu = -\mu_0 \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu J^\nu$$

multiply by $(\Lambda^\mu{}_\nu)^{-1}$ to get $\Box A^\mu = -\mu_0 J^\mu$ ✓

(? How about the Lorenz gauge condition — how does that respond to a Lorentz boost?)

Good — we know the eqn. we need to solve —

$$\Box A^\mu = -\mu_0 J^\mu$$

four copies of the same thing —

$$\Box h(\vec{r},t) = \underset{\text{const.}}{-k}\, \underset{\text{given}}{s(\vec{r},t)} \qquad (+)$$

How do we solve this eqn.?

<u>Green's Function Solution of PDE's</u>

A "Green's Function" is the solution to the "point source" problem.

Suppose $h(\vec{r},t) = h(\vec{r})$ (static), then $(+)$ is: (source is static)

$$\nabla^2 h(\vec{r}) = -k\, s(\vec{r}) \qquad (\Delta)$$

the Green's function problem is:

$$\nabla^2 G(\vec{r},\vec{r}\,') = -k\, \underset{\text{pt. src. at } \vec{r}\,'}{\delta^3(\vec{r}-\vec{r}\,')} \qquad (o)$$

If you can solve (o) for $G(\vec{r},\vec{r}\,')$, then

$$h(\vec{r}) = \int_{\text{all space}} G(\vec{r},\vec{r}\,')\, s(\vec{r}\,')\, d\tau'$$

solves (Δ) - proof:

$$\nabla^2 h(\vec{r}) = \int_{\text{all space}} [\nabla^2 G(\vec{r},\vec{r}\,')]\, s(\vec{r}\,')\, d\tau'$$

$$= \int_{\text{all space}} [-h\,\delta^3(\vec{r}-\vec{r}\,')]\, s(\vec{r}\,')\, d\tau'$$

since (o) is satisfied by assumption.

$$= -h\, s(\vec{r}) \checkmark$$

using the $\delta^3(\vec{r}-\vec{r}\,')$ to carry out the integral.

Okay - but how do we solve (o) ?

1. Set the pt. src. to the origin: $\vec{r}\,' \to 0$ (we'll move it back later)

$$\nabla^2 G(\vec{r}) = -h\,\delta^3(\vec{r})$$

The point source is spherically symmetric, "so" $G(\vec{r})$ is, too (?)

assume $G(\vec{r}) = G(r)$.

2. Solve $\nabla^2 G(r) = -h\,\delta^3(\vec{r})$ at locations away from the source, i.e. at points where $r \neq 0$:

$$\nabla^2 G(r) = \frac{1}{r}(rG)'' \quad \left(\text{primes denote } \frac{d}{dr}\right)$$

so $\frac{1}{r}(rG)'' = 0 \Rightarrow (rG)'' = 0$

then $(rG)' = B \longrightarrow rG = Br + A$

$\to G(r) = \frac{A}{r} + B$ + we'll take the b.c. $G \xrightarrow[r\to\infty]{} 0$

$\Rightarrow B = 0$

3. Fix A by integrating over a ball of radius ε centered at the origin

$$\int_{B(0,\epsilon)} \nabla\cdot(\nabla G)\, d\tau = -h\int_{B(0,\epsilon)} \delta^3(\vec{r})\, d\tau$$

div. thm.

$$\int_{\partial B} \nabla G \cdot d\vec{a} = -h$$

+ this gives: $4\pi\epsilon^2 \cdot \left(\frac{-A}{\epsilon^2}\right) = -4\pi A = -h \Rightarrow A = \frac{h}{4\pi}$

$$G(r) = \frac{h}{4\pi r}$$

4. set the source back to $\vec{r}\,'$

$$G(\vec{r},\vec{r}\,') = \frac{h}{4\pi|\vec{r}-\vec{r}\,'|}$$ which looks... familiar.