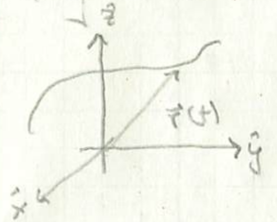
Last Time

For a particle moving along $\vec{r}(t)$



you can always construct a boost into the particle's instantaneous rest frame, using

$$v(t) = \sqrt{\dot{\vec{r}}(t)\cdot\dot{\vec{r}}(t)}$$

as the boost speed.

proper time is the time measured in that instantaneous rest frame:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}$$

Using τ we made 4-vectors:

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}\begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$p^\mu = m\,\eta^\mu = \frac{1}{\sqrt{1-v^2/c^2}}\begin{pmatrix} mc \\ m\vec{v} \end{pmatrix}$$

rest mass

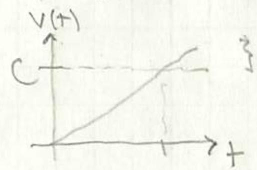
Newton's 2nd Law

Recall the issue w/:

$$\frac{dp}{dt} = F \quad \text{w/ } p = mv$$

for a constant force $\vec{F} = \vec{F}_0$, we have

$$p(t) = F_0 t \Rightarrow v(t) = F_0 t/m$$



$\} > c \ldots$

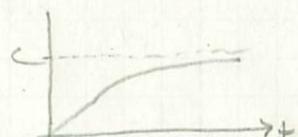
What if we instead take:

$$p = \frac{mv}{\sqrt{1-v^2/c^2}} = m\frac{dx}{d\tau}$$

the relativistic momentum. Now:

$$p^2(1-v^2/c^2) = m^2v^2 \Rightarrow v = \frac{p/m}{\sqrt{1+\left(\frac{p}{mc}\right)^2}}$$

w/



p increases to ∞, but

$$v \xrightarrow[p\to\infty]{} c$$

What about $\frac{d}{d\tau}p = F$ what is that?

Just as $m\ddot{x} = -\frac{dU}{dx} \Rightarrow E = \frac{1}{2}m\dot{x}^2 + U(x)$ in non-relativistic mechanics, we have

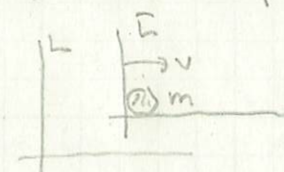
$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-v^2/c^2}}\right) = -\frac{dU}{dx} \Rightarrow E = \frac{mc^2}{\sqrt{1-v^2/c^2}} + U(x)$$

for conservation of energy in relativistic mechanics.

Energy-Momentum 4-vector

$p^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$ is a 4-vector by construction —

let's check it explicitly for a simple case:



m at rest in $\bar{L}$ has

$$\bar{E} = mc^2 \qquad \bar{p}_x = 0$$

in L: $E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \qquad p_x = \frac{mv}{\sqrt{1-v^2/c^2}}$

? $\quad E/c \overset{?}{=} \gamma(\bar{E}/c + v/c\,\bar{p}_x)$

$\quad p_x \overset{?}{=} \gamma(v/c\,\bar{E}/c + \bar{p}_x)$

well: $E/c = \frac{mc}{\sqrt{1-v^2/c^2}}$ & $\gamma(\bar{E}/c + v/c\,\cancel{\bar{p}_x}) = \frac{mc}{\sqrt{1-v^2/c^2}}$ ✓

$p_x = \frac{mv}{\sqrt{1-v^2/c^2}}$ & $\gamma(v/c\,\bar{E}/c + \cancel{\bar{p}_x}) = \frac{mv}{\sqrt{1-v^2/c^2}}$ ✓

Just as $-c^2 d\bar{t}^2 + d\bar{x}^2 = -c^2 dt^2 + dx^2$ for the coordinates, we have:

$$-\bar{E}^2/c^2 + \bar{p}_x^2 = -E^2/c^2 + p_x^2$$

check: $-\bar{E}^2/c^2 + \bar{p}_x^2 = -m^2c^2$

$$-E^2/c^2 + p_x^2 = \frac{-m^2c^2}{(1-v^2/c^2)} + \frac{m^2v^2}{(1-v^2/c^2)} = \frac{-m^2c^2(1-v^2/c^2)}{(1-v^2/c^2)} = -m^2c^2 \checkmark$$

we can always boost to the instantaneous rest frame, so

$$-E^2/c^2 + p^2 = -m^2c^2$$

Importance of 4-Vectors

"The laws of physics are the same in all (inertial) reference frames"

If you can state the "laws of physics" using 4-vectors, then this statement will hold.

law of physics in $\bar{L}$: $\bar{A}^\mu = \bar{B}^\mu$ for 4-vecs A^μ, B^μ:

$$\sum_{\nu=0}^{3} \Lambda^\mu{}_\nu A^\nu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu B^\nu \leftarrow \text{for } \mu = 0,1,2,3$$

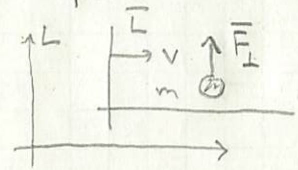
mult. by Λ^{-1} to get $A^\mu = B^\mu$ the law holds in L.

? what sets of 4-things w/ the same units can you think of? ✓

can use c to convert.

Force Transformation

Forces have 3 vector components. How do these respond to a Lorentz boost?



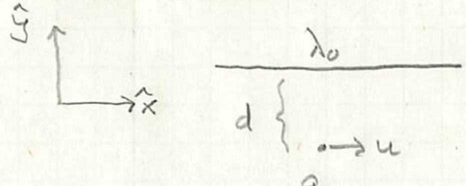
in $\bar{L}$: $\bar{F}_\perp = \frac{d\bar{p}_\perp}{d\bar{t}}$

in L: $F_\perp = \frac{dp_\perp}{dt}$

& $d\bar{p}_\perp = dp_\perp$, $c\,dt = \gamma(c\,d\bar{t} + v/c\,\cancel{d\bar{x}})$ → $dt = \gamma\, d\bar{t}$

so $F_\perp = \frac{dp_\perp}{dt} = \frac{d\bar{p}_\perp}{\gamma\, d\bar{t}} = \frac{1}{\gamma}\bar{F}_\perp$.

Model Problem

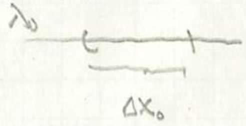


infinite line of charge w/ uniform λ, at rest.

? What is the force on q?

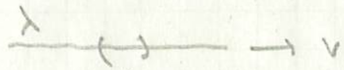
$\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0 d}(-\hat{y})$ w/ $\vec{F} = q\vec{E}$.

For the line of charge at rest:



the charge enclosed in a Δx_0 segment is: $q = \lambda_0 \Delta x_0$

If the line of charge moves to the right w/ speed v:



length contraction: $\Delta x = \frac{1}{\gamma}\Delta x_0$

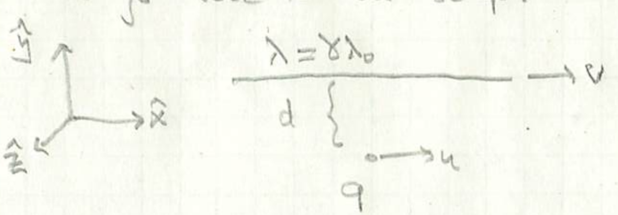
$q = \lambda \Delta x = \lambda_0 \Delta x_0$ (same charge q in both cases)

$\frac{\lambda}{\gamma}\Delta x_0 = \lambda_0 \Delta x_0$

so $\lambda = \gamma \lambda_0 > \lambda_0$

makes sense - the same amount of charge in a small segment, so the density is larger.

let's go back to our setup:



? what is the force on q?

We now have $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 d}(-\hat{y})$ & $\vec{B} = \frac{\mu_0 \lambda v}{2\pi d}(-\hat{z})$

&

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

$$= \frac{q\lambda_0\gamma}{2\pi\epsilon_0 d}(-\hat{y}) + q\frac{\mu_0\lambda_0\gamma uv}{2\pi d}(\hat{y})$$

$$= -\frac{q\lambda_0}{2\pi\epsilon_0 d}\left[1 - \mu_0\epsilon_0 uv\right]\hat{y}$$

$$= -\frac{q\lambda_0}{2\pi\epsilon_0 d}\left[1 - \frac{uv}{c^2}\right]\hat{y}$$