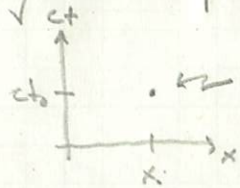
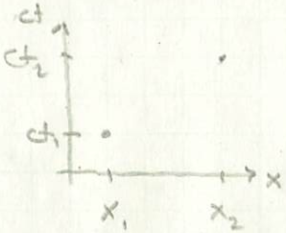
Minkowski Diagrams & Causality

A Minkowski diagram is a plot of points in the x–ct plane.



← an "event" at location x_0, time t_0.

An event at x_1, t_1 can *cause* an event at x_2, t_2 if a signal can travel from x_1 to x_2 at a speed $\leq c$.



$$\frac{x_2 - x_1}{t_2 - t_1} \leq c$$

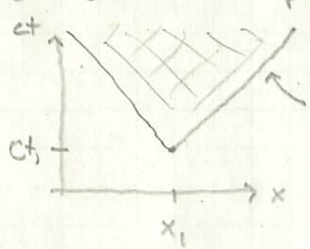
or $(x_2 - x_1) \leq c(t_2 - t_1)$

the requirement, written in terms of the Minkowski distance (squared) is

$$-c^2(t_2-t_1)^2 + (x_2-x_1)^2 \leq 0$$

(such an interval is called "time-like")

Given a point x_1, t_1, the region of "causal influence" is the set of all points in causal contact w/ x_1, t_1:



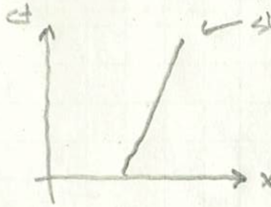
← 45° "light" line.

In D = 3+1, this triangle becomes a cone, the "light cone"

Motion

For a particle traveling at speed v: $x = vt$, or

$$ct = \frac{c}{v} x$$



← slope is c/v

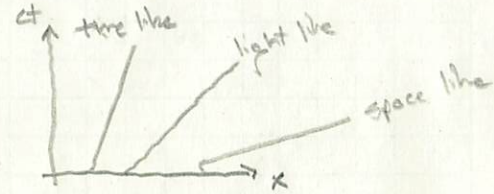
lines w/ slope > 1 have

$$c/v > 1 \Rightarrow v < c$$

particle motion occurs w/ $v < c$.

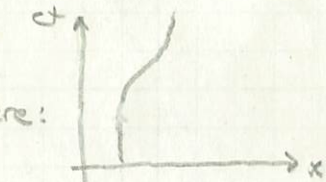
For light: $x = ct \Rightarrow ct = x$, the slope is 1

Slopes < 1 have $v > c$



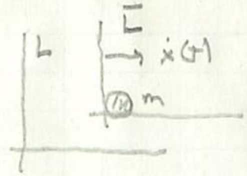
for non-constant vel. motion:

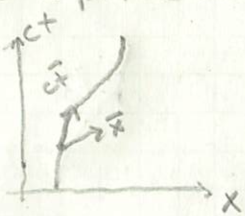
? tell the "story" of motion here:



Proper Time

Given a curve $x(t)$, at any t, we can make a Lorentz boost into the "rest-frame" of a particle moving according to $x(t)$:





In L: $ds^2 = -c^2dt^2 + dx^2$, for a small interval

In $\bar{L}$: $d\bar{s}^2 = -c^2d\bar{t}^2 + \cancel{d\bar{x}^2}$ rest frame has $d\bar{x}=0$.

But $d\bar{s}^2 = ds^2 \Rightarrow -c^2d\bar{t}^2 = -c^2dt^2 + dx^2$

$$= -c^2dt^2 + \dot{x}(t)^2dt^2$$

the relation between t & $\bar{t}$ is given by

$$d\bar{t}^2 = dt^2(1-\dot{x}(t)^2/c^2) \Rightarrow \frac{dt}{d\bar{t}} = \frac{1}{\sqrt{1-\dot{x}(t)^2/c^2}}$$

in this context $d\bar{t}$ is a "proper time" interval, w/ $\bar{t}$ defined in the instantaneous rest frame of the particle – often denoted τ:

$$\boxed{\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\dot{x}(t)^2/c^2}}} = \frac{1}{\sqrt{1-\vec{v}\cdot\vec{v}/c^2}}$$

in D=3
w/ $\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$

Lorentz Vectors

For a Lorentz boost in the x-direction:

$$\underbrace{\begin{pmatrix} c\,d\bar{t} \\ d\bar{x} \\ d\bar{y} \\ d\bar{z} \end{pmatrix}}_{d\bar{x}^\mu} = \begin{pmatrix} \gamma & -v/c\,\gamma & 0 & 0 \\ -v/c\,\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix} \quad (*)$$

$$d\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu\, dx^\nu \quad \text{for } \mu = 0, 1,2,3 \quad (+)$$

time; space

we call dx^μ a "4-vector" – anything else w/ 4-components, (all w/ the same units) that responds to a Lorentz transformation like (*) is also called a 4-vector.

<u>Example</u> $\eta^\mu = \frac{dx^\mu}{d\tau}$ is a 4-vector

to see this, take carry out a Lorentz transformation:

$$\bar{\eta}^\mu = \frac{d\bar{x}^\mu}{d\bar{\tau}}$$

& note that $d\bar{\tau} = d\tau$, the <u>proper</u> time elapsed is unchanged by the boost.

$$= \frac{d\bar{x}^\mu}{d\tau} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \frac{dx^\nu}{d\tau} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu\, \eta^\nu \quad \text{for } \mu = 0,1,2,3$$

matching (+) (which is equivalent to (*))

$\eta^\mu = \frac{dx^\mu}{d\tau}$ is the "4-velocity"

w/ components: $\eta^0 = c\frac{dt}{d\tau}$, $\eta^i = \frac{dx^i}{d\tau}$ $i = 1,2,3$

How is η^μ related to $\frac{dx^i}{dt} \equiv v^i$ $(i=1,2,3)$?

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} \doteq \frac{1}{\sqrt{1-v^2/c^2}} \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

When ... "updating" non-relativistic expressions to take SR into account, we have to decide whether to use t or τ – they are the same for $v \ll c$.

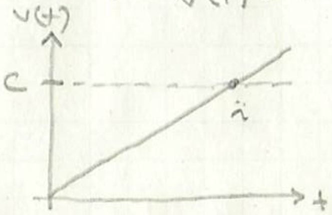
Newton's 2nd Law

Working in one-dimension, Newton's 2nd law is:

$$\frac{dp}{dt} = F \quad \text{w/ } p = mv \qquad (*)$$

issue: for $F = F_0$, constant, $p(t) = F_0 t$ & then

$$v(t) = p(t)/m = F_0/m \; t$$



How should we update $(*)$ to bring the predicted motion in line w/ SR?

$$\frac{dp}{dt} = F \quad \text{w/ } p = m\eta = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

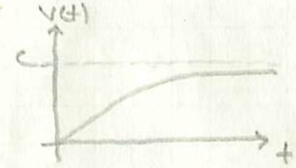
we still have $p(t) = F_0 t$ for constant force, but:

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \Rightarrow p^2(1 - v^2/c^2) = m^2 v^2$$

$$p^2 = v^2(m^2 + p^2/c^2)$$

so $$v = \frac{p/m}{\sqrt{1 + (p/mc)^2}} \xrightarrow[p \to \infty]{} c \checkmark$$

now:



starts off w/ $v \approx F_0/m \, t$ (linear), but asymptotically approaches c.

=

$p^\mu \equiv m \frac{dx^\mu}{d\tau}$ is the "relativistic 4-momentum" it is a 4-vector.

the spatial components are $\vec{p} = m\vec{\eta} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$

How about p^0? $$p^0 = mc\frac{dt}{d\tau} = \frac{mc}{\sqrt{1 - v^2/c^2}} = \frac{1}{c}\left[\frac{mc^2}{\sqrt{1 - v^2/c^2}}\right]$$

$p^0 = E/c$

all 4 components of p^μ have units of momentum.

? Any other sets of 4 things w/ the same units (allowing only c to convert, as in p^μ)?

? Why isn't Newton's 2nd law: $\frac{dp}{d\tau} = F$ in SR?