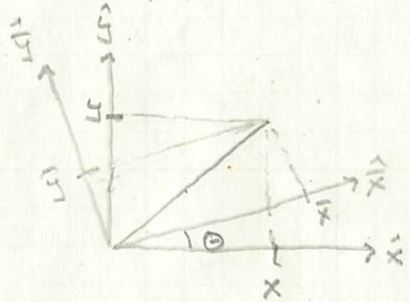
Rotations

A coordinate transformation is often designed to leave some quantity unchanged.



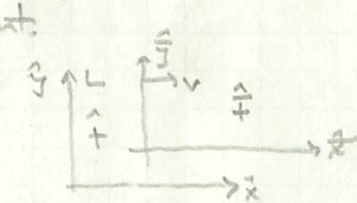
$$\bar x = x\cos\theta + y\sin\theta$$
$$\bar y = -x\sin\theta + y\cos\theta$$
w/ $\bar x^2 + \bar y^2 = x^2 + y^2$

lengths are unchanged.

Rotations are the simplest (linear) transformations that leave lengths unchanged.

The relation $\cos^2\theta + \sin^2\theta = 1$ what enforces the invariant.

In SR:



} $\hat{\bar x} = \hat x$ - they are offset for clarity

If the origins coincide at $t = \bar t = 0$ & a light flashes on & off at that point, the x-loc of the light in L is: $x = ct$

in $\bar L$: $\bar x = c\bar t$ } speed of light is the same in L & $\bar L$

Lorentz Boost

$x = ct \Rightarrow -c^2t^2 + x^2 = 0$ in L, &

$\bar x = c\bar t \Rightarrow -c^2\bar t^2 + \bar x^2 = 0$ in $\bar L$, so $-c^2\bar t^2 + \bar x^2 = -c^2t^2 + x^2$

What is the simplest, linear, transformation that enforces this?

Use: $\cosh^2\eta - \sinh^2\eta = 1$ $\quad (e^\eta = \cosh\eta + \sinh\eta)$

$$c\bar t = ct\cosh\eta - x\sinh\eta \qquad \bar x = -ct\sinh\eta + x\cosh\eta$$

This will do it — but how do cosh & sinh relate to the physical setup, w/ L & $\bar L$?

Pick a point - the origin of $\bar L$, w/ $\bar x = 0$ & $x = vt$. Then the transformation gives:

$$c\bar t = ct\cosh\eta - vt\sinh\eta \qquad 0 = -ct\sinh\eta + vt\cosh\eta$$

or $\frac{\sinh\eta}{\cosh\eta} = \frac{v}{c} = \tanh\eta$

From this relation, we have: $\sinh^2\eta = v^2/c^2\cosh^2\eta$
$= v^2/c^2(1 + \sinh^2\eta)$

&

$$\sinh\eta = \frac{v/c}{\sqrt{1 - v^2/c^2}} \qquad \cosh\eta = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$

so $c\bar t = \gamma(ct - v/c\, x)$ $\quad \bar x = \gamma(-v/c\, ct + x)$

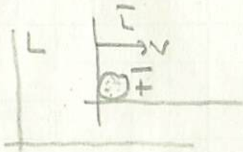
since from $\bar L$'s p.o.v., L moves to the left w/ speed v:

$$ct = \gamma(c\bar t + v/c\,\bar x) \qquad x = \gamma(v\bar t + \bar x)$$

is the "inverse" transformation.

Time Dilation

A clock is at rest in $\bar{L}$, sitting at the origin ($\bar{x}=0$). For a time $\bar{t}$ ticked off by the clock in $\bar{L}$, how much time elapses for a clock at rest in L?



approach #1: use $-c^2\bar{t}^2+\bar{x}^2=-c^2t^2+x^2$ directly, w/ $\bar{x}=0$, $x=vt$:

$$-c^2\bar{t}^2=-c^2t^2+v^2t^2 \Rightarrow \bar{t}^2=(1-v^2/c^2)t^2$$

$$t=\frac{1}{\sqrt{1-v^2/c^2}}\bar{t}=\gamma\bar{t}$$

approach #2: use $ct=\gamma(+v/c\,\cancel{\bar{x}}+c\bar{t})$

$$t=\gamma\bar{t} \checkmark$$

for $v=4/5\,c$, $1-v^2/c^2=1-16/25=9/25$

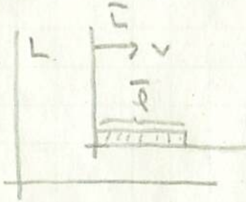
$$t=\frac{1}{\sqrt{1-v^2/c^2}}\bar{t}=5/3\,\bar{t}$$

so if $\bar{t}=1$ year passes in $\bar{L}$, $t=5/3$ year passes in L.

The time elapsed in L is <u>longer</u> than that in $\bar{L}$.

Length Contraction

A stick is at rest in $\bar{L}$ where it has length $\bar{l}$. What is the length of the stick in L?



when you want to measure length, you measure instantaneously so that in L, $\Delta t=0$, so we want Δx

Using the Lorentz boost:

$$\underbrace{\Delta\bar{x}}_{\bar{l}}=\gamma(-v\underbrace{\Delta t}_{=0\text{ for a length in }L}+\Delta x)$$

we get: $\Delta x=\frac{1}{\gamma}\Delta\bar{x}=\sqrt{1-v^2/c^2}\cdot\bar{l}$

so that if $v=4/5\,c$, $l=3/5\,\bar{l}<\bar{l}$.

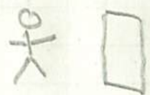
A meterstick in $\bar{L}$ is a 3/5 m stick in L.

Lorentz Transformation

How about $\bar{y}$ & $\bar{z}$, how are they related to $\{ct, x, y, z\}$?

→ <u>Suppose</u> $\hat{y}$ experienced length contraction, as $\hat{x}$ does.

Consider:



$\}h$ both me & a door have height h at rest.

I run towards the door at speed v. From the door's p.o.v., I have height $< h$, & so make it through the door.

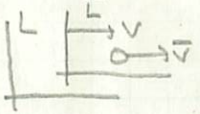
From my p.o.v. the door is moving towards me at speed v, so it is shorter, & I don't make it through.

In order for the door & I to make the same prediction, there must be no length contraction in the y or z directions.

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Velocity Addition

pre-special-relativity: $\bar{L}$ moves w/ speed v relative to L, & a ball moves at speed $\bar{v}$ relative to $\bar{L}$ - the ball's speed relative to L is:



$$v_L = v + \bar{v}$$

* Issue: If $v = c$ & $\bar{v} = c$, $v_L = 2c$

but if something moves at c in L, it moves at c in $\bar{L}$, &

$$2c \neq c \quad \ldots$$

Special Relativity: the ball moves $\Delta\bar{x} = \bar{v}\,\Delta\bar{t}$ in $\bar{L}$ & it has $\Delta x = v_L \Delta t$ in L, so

$$v_L = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta\bar{x} + \frac{v}{c} \cdot c\Delta\bar{t})}{\frac{1}{c}\gamma(\frac{v}{c}\Delta\bar{x} + c\Delta\bar{t})}$$ using the Lorentz boost relation

$$= \frac{\Delta\bar{t}(\bar{v} + v)}{\Delta\bar{t}(1 + \frac{v}{c^2}\frac{\Delta\bar{x}}{\Delta\bar{t}})} = \frac{v + \bar{v}}{1 + v\bar{v}/c^2}$$

For $v \ll c$, $\bar{v} \ll c$, $v_L \approx v + \bar{v}$ ✓

For $v = c$, $\bar{v} = c$, $v_L = \frac{2c}{1 + c^2/c^2} = c$ ✓