Presentation Problem 1

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Problem (as given in assignment)

We discussed the gradient operator:

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

and identified its action on a scalar function as indicating the direction of greatest increase. Use this observation to generate the basis vector in spherical coordinates.

Presentation

In this problem, we are asked to use the gradient operator, expressed in Cartesian coordinates,

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{1}$$

to develop the unit basis vectors in spherical coordinates, $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}\}$.

The gradient is a useful tool for this problem because when it is applied to a scalar function like $f(x, y, z)$, it returns the direction of greatest increase of f. That characteristic feature of the gradient can be established quickly using Taylor expansion: For a point $\{x, y, z\}$, we know the value of f. At a nearby point, displaced from the original by (small) dx, dy and dz, the value of f is approximately:

$$f(x + dx, y + dy, z + dz) \approx f(x, y, z) + dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y} + dz\frac{\partial f}{\partial z}, \tag{2}$$

or, defining $d\boldsymbol{\ell} \equiv dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$,

$$f(x + dx, y + dy, z + dz) \approx f(x, y, z) + d\boldsymbol{\ell} \cdot \nabla f(x, y, z), \tag{3}$$

and the way to maximize the dot product is to take $d\boldsymbol{\ell} \parallel \nabla f(x, y, z)$.

In any coordinate system, the basis vectors point in the direction of increasing coordinate value. For the function $r(x, y, z) \equiv \sqrt{x^2 + y^2 + z^2}$ that defines the radial coordinate, the direction of greatest increase, at $\{x, y, z\}$, is $\nabla r(x, y, z)$, all we have to do is calculate the derivatives:

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2} 2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \tag{4}$$

with the y and z derivatives obtained by taking $x \to y$ and $x \to z$ respectively. The gradient of r is then

$$\nabla r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \right]. \tag{5}$$

We want $\hat{\mathbf{r}}$, the *unit* vector pointing in the direction of greatest increase for the coordinate r, but it is clear that $\nabla r(x, y, z)$ above already has $\nabla r(x, y, z) \cdot \nabla r(x, y, z) = 1$, so we have

$$\hat{\mathbf{r}} = \nabla r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \right]. \tag{6}$$

Moving on to the angular pieces, we have $\phi = \tan^{-1}(y/x)$. Our first job is to identify the derivative of the arctangent. One useful trick is to note that

$$\psi = \tan^{-1}(\tan \psi), \tag{7}$$

then let $u \equiv \tan \psi$.[1] Taking the u-derivative of the equation that defines u gives

$$1 = \frac{d \tan \psi}{d\psi} \frac{d\psi}{du} \longrightarrow \frac{d\psi}{du} = \cos^2 \psi = \frac{1}{1 + u^2}. \tag{8}$$

Now taking the u derivative of both sides of (7), we have

$$\frac{d\psi}{du} = \frac{d \tan^{-1}(u)}{du} = \frac{1}{1 + u^2}. \tag{9}$$

Returning to ϕ, the x and y derivatives are

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2} \qquad \frac{\partial \phi}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}. \tag{10}$$

The gradient of ϕ is, then,

$$\nabla \phi = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2 + y^2}. \tag{11}$$

[1] Inverting $u \equiv \tan \psi$ can be achieved by noting that $u^2 = (1 - \cos^2 \psi)/\cos^2 \psi$ and isolating $\cos^2 \psi = 1/(1 + u^2)$.

Normalizing to get $\hat{\boldsymbol{\phi}}$ gives

$$\nabla\phi \parallel \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2+y^2}}. \tag{12}$$

Finally, for $\theta \equiv \tan^{-1}(\sqrt{x^2+y^2}/z)$, we just have more application of the chain rule. The derivatives are

$$\frac{\partial\theta}{\partial x} = \frac{xz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} \qquad \frac{\partial\theta}{\partial y} = \frac{yz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} \qquad \frac{\partial\theta}{\partial z} = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2} \tag{13}$$

and the gradient is

$$\nabla\theta = \frac{xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - (x^2+y^2)\hat{\mathbf{z}}}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}. \tag{14}$$

Normalizing gives

$$\nabla\theta \parallel \hat{\boldsymbol{\theta}} = \frac{xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - (x^2+y^2)\hat{\mathbf{z}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}. \tag{15}$$

None of these expressions for the spherical basis vectors look at all familiar, but that is because they are written in Cartesian coordinates. Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, we recover

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}. \end{aligned} \tag{16}$$

It is interesting to note that the gradient's role in determining directions of increase holds even in spaces that are not "flat" (ones that have a different definition of length than the usual Pythagorean one). How might we determine the relevant unit basis vectors in a setting that was not just a coordinate transformation away from Cartesian?