Problem 1 | Problem Set 3 (Assignment 5)

For $\vec{A} = \frac{-y\hat{x} + x\hat{y}}{x^2+y^2}$ we have, in cylindrical coordinates, $\vec{A} = \frac{-\sin\phi\hat{x} + \cos\phi\hat{y}}{s}$

or $\vec{A} = \frac{\hat{\phi}}{s}$, w/ $\nabla\times\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(s\cdot\frac{1}{s})\hat{z} = 0$, except, potentially, at the origin.

Stokes' theorem says $\int_S (\nabla\times\vec{A})\cdot d\vec{a} = \oint_{\partial S} \vec{A}\cdot d\vec{l}$

Take the surface to be a disk of radius R, w/ $d\vec{a} = s\,d\phi\,ds\,\hat{z}$ & $d\vec{l} = R\,d\phi\,\hat{\phi}$ along the boundary. Then:

$$\oint_{\partial S} \vec{A}\cdot d\vec{l} = \int_0^{2\pi} \frac{1}{R}\cdot R\,d\phi = 2\pi$$

& in order to get $\int_S (\nabla\times\vec{A})\cdot d\vec{a} = 2\pi$, we must have $\nabla\times\vec{A} = 2\pi\delta^2(x,y)\hat{z}$,

$$\nabla\times\left(\frac{1}{s}\hat{\phi}\right) = 2\pi\delta^2(x,y)\hat{z}$$

Problem 2

We have (from the definition): $1 = \int_{-\infty}^{+\infty}\delta(x)\,dx = \int_{-\infty}^{0}\delta(x)\,dx + \int_0^{\infty}\delta(x)\,dx$

set $y=-x$ in the 1st integral

$$= -\int_{\infty}^{0}\underbrace{\delta(-y)}_{\delta(y)}\,dy + \int_0^{\infty}\delta(x)\,dx$$

$$= \int_0^{\infty}\delta(y)\,dy + \int_0^{\infty}\delta(x)\,dx = 2\int_0^{\infty}\delta(x)\,dx$$

so $\int_0^{\infty}\delta(x)\,dx = 1/2$

Problem 3 (1.46)

a. $\int_{-\infty}^{+\infty} \underbrace{f(x)}_{\text{"u"}}\left[x\underbrace{\frac{d}{dx}\delta(x)}_{\text{"v'"}}\right] \overset{\text{int. by parts}}{=} \underbrace{f(x)\cdot x\cdot\delta(x)\Big|_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty}\delta(x)\frac{d}{dx}(x f(x))\,dx$

$$= -\int_{-\infty}^{+\infty}\delta(x)(f(x) + x f'(x))\,dx = -f(0)$$

so $x\frac{d}{dx}\delta(x) = -\delta(x)$

b. $\int_{-\infty}^{+\infty}\underbrace{f(x)}_{\text{"u"}}\underbrace{\frac{d\Theta}{dx}}_{\text{"v'"}}\,dx = f(x)\Theta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\Theta(x)\frac{df}{dx}\,dx$

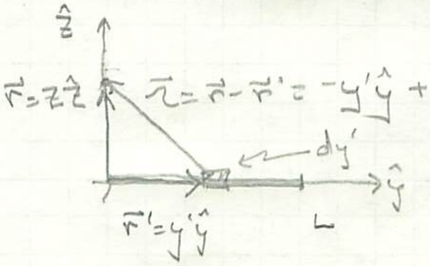
$$= f(\infty) - \int_0^{\infty}\frac{df}{dx}\,dx = f(\infty) - (f(\infty) - f(0))$$

$$= f(0) \qquad \text{so } \frac{d\Theta(x)}{dx} = \delta(x)$$

In order to get $\int_0^{\infty}\delta(x)\,dx = 1/2$, we must have $\int_0^{\infty}\frac{d\Theta}{dx}\,dx = 1/2$

$$\underbrace{\Theta(\infty)}_{=1} - \Theta(0) = 1/2 \Rightarrow \Theta(0) = 1/2$$

Problem 1 (Griffiths 2.3)



$\vec{r} = z\hat{z}$, $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -y'\hat{y} + z\hat{z}$, $\vec{r}' = y'\hat{y}$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{(-y'\hat{y} + z\hat{z})}{(y'^2+z^2)^{3/2}}\,dy'$$

& $\int \frac{-y'}{(y'^2+z^2)^{3/2}}dy' = \frac{1}{\sqrt{y'^2+z^2}}$, $\int \frac{dy'}{(y'^2+z^2)^{3/2}} = \frac{y'/z^2}{\sqrt{y'^2+z^2}}$

so
$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{y'^2+z^2}}\Big|_{y'=0}^{L}\hat{y} + \frac{y'/z}{\sqrt{y'^2+z^2}}\Big|_{y'=0}^{L}\hat{z}\right]$$
$$= \frac{\lambda}{4\pi\epsilon_0}\left[\left(\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right)\hat{y} + \frac{L/z}{\sqrt{L^2+z^2}}\hat{z}\right]$$

& $\lambda = Q/L$, we can write $\vec{E}$ as:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 L}\cdot\left[\left(\frac{1}{z\sqrt{1+(L/z)^2}} - \frac{1}{z}\right)\hat{y} + \frac{L/z}{z\sqrt{1+(L/z)^2}}\hat{z}\right]$$

so as $(L/z)^2 \to 0$, $\vec{E} \to \frac{Q}{4\pi\epsilon_0 z^2}\hat{z}$, the field of a point charge at the origin ✓.

Problem 2 (Griffiths 2.4)

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{\lambda a}{\sqrt{z^2+(a/2)^2}\sqrt{z^2+(a/2)^2+(a/2)^2}}\left[\cos\theta\,\hat{z} + \sin\theta\,\hat{y}\right]$$

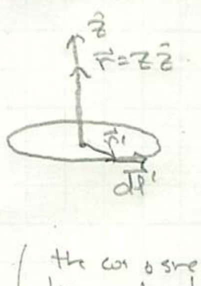
(using $\vec{E} = \frac{2\lambda L}{4\pi\epsilon_0}\cdot\frac{1}{z\sqrt{z^2+L^2}}\hat{z}$ w/ $z \to \sqrt{z^2+(a/2)^2}$, $L \to a/2$
& $\hat{z} \to \cos\theta\,\hat{z} + \sin\theta\,\hat{z}$)

& Similarly for the other sides - only the $\hat{z}$ component will be nonzero, so we get:

$\cos\theta = z/(z^2+(a/2)^2)^{1/2}$

$$\vec{E} = \frac{4\lambda a\cos\theta\,\hat{z}}{4\pi\epsilon_0\sqrt{(z^2+(a/2)^2)(z^2+(a/2)^2+(a/2)^2)}} = \frac{\lambda a z}{\pi\epsilon_0(z^2+(a/2)^2)\sqrt{z^2+a^2/2}}\hat{z}$$

Problem 3 (Griffiths 2.5)



$\vec{r} = z\hat{z}$ $\vec{r}' = r\cos\phi'\hat{x} + r\sin\phi'\hat{y}$, $dl' = r\,d\phi'$

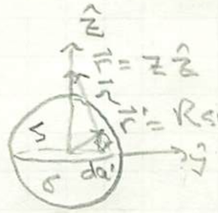
$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -r\cos\phi'\hat{x} - r\sin\phi'\hat{y} + z\hat{z}$ w/ $\mathscr{r} = (r^2+z^2)$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_0^{2\pi}\frac{(-r\cos\phi'\hat{x} - r\sin\phi'\hat{y} + z\hat{z})}{(r^2+z^2)^{3/2}}\cdot r\,d\phi'$$

(the cos & sine terms integrate to zero ✓)

$$= \frac{\lambda}{4\pi\epsilon_0}\frac{2\pi r z\hat{z}}{(r^2+z^2)^{3/2}} = \frac{\lambda r z}{2\epsilon_0(r^2+z^2)^{3/2}}\hat{z}$$

Problem 1 (Griffiths 2.7) | Problem Set 3 (Assignment 7)



$$da' = R^2 \sin\theta' \, d\theta' \, d\phi'$$

$$\vec r = z\hat z$$

$$\vec r' = R\sin\theta'\cos\phi'\,\hat x + R\sin\theta'\sin\phi'\,\hat y + R\cos\theta'\,\hat z$$

$$\vec{\mathscr{r}} = \vec r - \vec r' = -R\sin\theta'\cos\phi'\,\hat x - R\sin\theta'\sin\phi'\,\hat y + (-R\cos\theta' + z)\,\hat z$$

w/

$$\mathscr{r} = R^2 + z^2 - 2Rz\cos\theta'$$

then

$$\vec E = \oint_S \frac{\sigma(\vec r')\,\vec{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^3}\,da' = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{[-R\sin\theta'\cos\phi'\,\hat x - R\sin\theta'\sin\phi'\,\hat y + (z - R\cos\theta')\,\hat z]\,R^2\sin\theta'\,d\theta'\,d\phi'}{(R^2+z^2-2Rz\cos\theta')^{3/2}}$$

ϕ' integration gets rid of $\hat x$ & $\hat y$ terms

$$= \frac{\sigma}{4\pi\epsilon_0}\cdot 2\pi R^2\,\hat z\int_0^{\pi}\frac{(z - R\cos\theta')\sin\theta'\,d\theta'}{(R^2+z^2-2Rz\cos\theta')^{3/2}}$$

let $u \equiv R\cos\theta'$, then $du = -R\sin\theta'\,d\theta'$, &

$$= \frac{\sigma R}{2\epsilon_0}\,\hat z\int_{-R}^{R}\frac{(z-u)}{(R^2+z^2-2zu)^{3/2}}\,du$$

note that

$$\frac{d}{du}\left[\frac{1}{\sqrt{A+Bu}}\right] = \frac{-\frac12\cdot B}{(A+Bu)^{3/2}} \Rightarrow \int\frac{1}{(A+Bu)^{3/2}}\,du = \frac{-2/B}{\sqrt{A+Bu}}$$

&

$$\frac{d}{du}\left[\frac{u}{\sqrt{A+Bu}}\right] = \frac{1}{\sqrt{A+Bu}} - \frac{\frac12 B}{(A+Bu)^{3/2}} = \frac{A+\frac12 Bu}{(A+Bu)^{3/2}} \Rightarrow \int\frac{u}{(A+Bu)^{3/2}}\,du = \frac{2}{B}\left[-A\int\frac{du}{(A+Bu)^{3/2}} + \frac{u}{\sqrt{A+Bu}}\right]$$

$$= \frac{2u + 4A/B}{B\sqrt{A+Bu}}$$

w/ $A \equiv R^2 + z^2$ & $B = -2z$

$$\vec E = \frac{\sigma R}{2\epsilon_0}\,\hat z\left[\frac{1}{\sqrt{R^2+z^2-2zu}}\Bigg|_{u=-R}^{R} - \frac{(2u - 2(R^2+z^2)/z)}{\sqrt{R^2+z^2-2zu}\;(-2z)}\Bigg|_{u=-R}^{R}\right]$$

$$= \frac{\sigma R}{2\epsilon_0}\,\hat z\left[\frac{2z + 2u - 2R^2/z - 2z}{2z\sqrt{R^2+z^2-2zu}}\Bigg|_{u=-R}^{R}\right] = \frac{\sigma R}{2\epsilon_0}\,\hat z\left[\frac{uz - R^2}{z^2\sqrt{R^2+z^2-2zu}}\right]\Bigg|_{u=-R}^{R}$$

$$= \frac{\sigma R}{2\epsilon_0 z^2}\,\hat z\left[\frac{R(z-R)}{\sqrt{(z-R)^2}} + \frac{R(z+R)}{\sqrt{(z+R)^2}}\right] = \frac{\sigma R^2}{2\epsilon_0 z^2}\,\hat z\left[1 + \frac{(z-R)}{\sqrt{(z-R)^2}}\right]$$

if $z < R$, $\sqrt{(z-R)^2} = R - z$, & $\vec E_{in} = \frac{\sigma R^2}{2\epsilon_0 z^2}\,\hat z\,[1 - 1] = 0$

if $z > R$ $\sqrt{(z-R)^2} = z - R$ & $\vec E_{out} = \frac{2\sigma R^2}{2\epsilon_0 z^2}\,\hat z = \frac{q}{4\pi\epsilon_0 z^2}\,\hat z$ ✓

$\sigma = \frac{q}{4\pi R^2}$

Problem 2 (Griffiths 2.9)

a. For $\vec{E} = kr^3\hat{r}$, we have $\nabla\cdot\vec{E} = \rho/\epsilon_0 \Longrightarrow \rho = \epsilon_0\nabla\cdot\vec{E} = \epsilon_0\frac{1}{r^2}\frac{d}{dr}(kr^5)$

$$= 5k\epsilon_0 r^2$$

b. The total charge in the sphere is $Q = \int_\Omega \rho\, d\tau$

$$= \int_0^R\int_0^{2\pi}\int_0^{\pi} 5k\epsilon_0 r^2 \cdot r^2\sin\theta\, d\theta\, d\phi\, dr$$

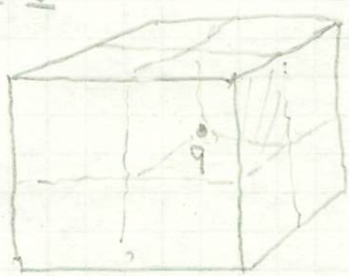
$$= 20\pi k\epsilon_0 \cdot \tfrac{1}{5}R^5 = 4\pi k\epsilon_0 R^5$$

We can also use $Q = \epsilon_0\oint \vec{E}\cdot d\vec{a}$, Gauss's Law

$$= \epsilon_0\int_0^{2\pi}\int_0^{\pi} kR^3\hat{r}\cdot(R^2\sin\theta\, d\theta\, d\phi\, \hat{r})$$

$$= 4\pi k\epsilon_0 R^5$$

Problem 3



For the large cube centered at the charge, the flux through any side is

$$\Phi_s = \frac{1}{\epsilon_0} q/6$$

& the desired shaded portion is 1/4 of the large cube side, so $\Phi = \frac{q}{24\epsilon_0}$.