Problem 1 (1.13c) | Problem Set 2

For $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$, the magnitude is:

$$\mathscr{r} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

& to compute $\nabla \mathscr{r}^n$, we can consider just the x-deriv. ($\mathscr{r}$ is symmetric in x, y & z):

$$\frac{\partial \mathscr{r}^n}{\partial x} = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x} = n\mathscr{r}^{n-1}\cdot\frac{\frac{1}{2}\cdot 2(x-x')}{\mathscr{r}} = n(x-x')\mathscr{r}^{n-2}$$

so
$$\nabla \mathscr{r}^n = \frac{\partial \mathscr{r}^n}{\partial x}\hat{x} + \frac{\partial \mathscr{r}^n}{\partial y}\hat{y} + \frac{\partial \mathscr{r}^n}{\partial z}\hat{z} = n\mathscr{r}^{n-2}\overbrace{\left[(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}\right]}^{=\vec{\mathscr{r}}}$$

check: for $n=2$, we have $\nabla \mathscr{r}^2 = 2\vec{\mathscr{r}}$ ✓

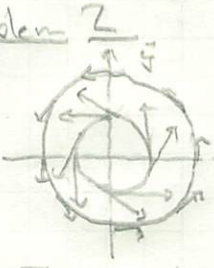
$n=-1$ $\qquad \nabla \mathscr{r}^{-1} = -\mathscr{r}^{-3}\vec{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ ✓

If we take primed derivs instead:

$$\frac{\partial \mathscr{r}}{\partial x'} = n\mathscr{r}^{n-1}\frac{\partial \mathscr{r}}{\partial x'} = n\mathscr{r}^{n-1}\cdot\frac{\frac{1}{2}\cdot 2(x-x')(-1)}{\mathscr{r}} = -n(x-x')\mathscr{r}^{n-2}$$

so $\qquad \nabla'\mathscr{r}^n = -n\mathscr{r}^{n-2}\vec{\mathscr{r}}$

Problem 2



$$\vec{A} = \frac{-y\hat{x} + x\hat{y}}{x^2+y^2} \quad \text{w/ } A = \left[\frac{y^2+x^2}{(x^2+y^2)^2}\right]^{1/2} = \frac{1}{\sqrt{x^2+y^2}}$$

length is smaller further away.

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = +\frac{y\cdot 2x}{(x^2+y^2)^2} - \frac{x\cdot 2y}{(x^2+y^2)^2} = 0$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/(x^2+y^2) & x/(x^2+y^2) & 0 \end{vmatrix} = \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]$$

$$= \hat{z}\left[\frac{1}{x^2+y^2} - \frac{x\cdot 2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y\cdot 2y}{(x^2+y^2)^2}\right]$$

$$= \hat{z}\left[\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2}\right] = 0$$

yes, I am surprised.

Problem 3

For $\vec{E} = \frac{E(r)}{r}\cdot\vec{r} = \frac{E(r)}{r}\left[x\hat{x} + y\hat{y} + z\hat{z}\right]$ w/ $r = (x^2+y^2+z^2)^{1/2}$

$\frac{\partial r}{\partial x} = x/r,\ \frac{\partial r}{\partial y} = y/r,\ \frac{\partial r}{\partial z} = z/r$

$$\nabla\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Ex/r & Ey/r & Ez/r \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{Ez}{r}\right) - \frac{\partial}{\partial z}\left(\frac{Ey}{r}\right)\right] - \hat{y}\left[\frac{\partial}{\partial x}\left(\frac{Ez}{r}\right) - \frac{\partial}{\partial z}\left(\frac{Ex}{r}\right)\right] + \hat{z}\left[\frac{\partial}{\partial x}\left(\frac{Ey}{r}\right) - \frac{\partial}{\partial y}\left(\frac{Ex}{r}\right)\right]$$

$$= \hat{x}\left[\frac{E'(r)\frac{\partial r}{\partial y}\cdot z}{r} - \frac{Ez}{r^2}\frac{\partial r}{\partial y} - \frac{E'(r)\cdot\frac{\partial r}{\partial z}y}{r} + \frac{Ey}{r^2}\cdot\frac{\partial r}{\partial z}\right]$$

$$- \hat{y}\left[\frac{E'(r)\frac{\partial r}{\partial x}z}{r} - \frac{Ez}{r^2}\frac{\partial r}{\partial x} - \frac{E'(r)\frac{\partial r}{\partial z}x}{r} + \frac{Ex}{r^2}\frac{\partial r}{\partial z}\right]$$

$$+ \hat{z}\left[\frac{E'(r)\frac{\partial r}{\partial x}y}{r} - \frac{Ey}{r^2}\frac{\partial r}{\partial x} - \frac{E'(r)\frac{\partial r}{\partial y}x}{r} + \frac{Ex}{r^2}\cdot\frac{\partial r}{\partial y}\right]$$

$$= \hat{x}\left[\frac{E'\cdot yz}{r^2} - \frac{Ezy}{r^3} - \frac{E'zy}{r^2} + \frac{Eyz}{r^3}\right]$$

$$- \hat{y}\left[\frac{E'xz}{r^2} - \frac{Ezx}{r^3} - \frac{E'xz}{r^2} + \frac{Exz}{r^3}\right]$$

$$+ \hat{z}\left[\frac{E'xy}{r^2} - \frac{Eyx}{r^3} - \frac{E'yx}{r^2} + \frac{Exy}{r^3}\right] = 0 \ \checkmark$$

$$\nabla\cdot\vec{E} = \frac{\partial}{\partial x}\left(\frac{Ex}{r}\right) + \frac{\partial}{\partial y}\left(\frac{Ey}{r}\right) + \frac{\partial}{\partial z}\left(\frac{Ez}{r}\right)$$

$$= \left[\frac{E}{r} + \frac{E'\frac{\partial r}{\partial x}x}{r} - \frac{Ex}{r^2}\frac{\partial r}{\partial x}\right] + \left[\frac{E}{r} + \frac{E'\frac{\partial r}{\partial y}y}{r} - \frac{Ey}{r^2}\frac{\partial r}{\partial y}\right] + \left[\frac{E}{r} + \frac{E'\frac{\partial r}{\partial z}z}{r} - \frac{Ez}{r^2}\frac{\partial z}{\partial r}\right]$$

$$= \frac{3E(r)}{r} + \frac{E'(r)}{r^2}\underbrace{\left(x^2+y^2+z^2\right)}_{=r^2} - \frac{E}{r^3}\underbrace{\left(x^2+y^2+z^2\right)}_{=r^2}$$

$$= \frac{3E(r)}{r} + E'(r) - \frac{E(r)}{r} = \frac{2E(r)}{r} + E'(r).$$

Problem 4

For $\vec{w}(s) = 2s\,\hat{x} + s^2\hat{y} + 2z\,\hat{z}$, so that, along the curve: $x=2s,\ y=s^2,\ z=2s$

$$d\vec{l} = \frac{d\vec{w}}{ds}ds = (2\hat{x} + 2s\hat{y} + 2\hat{z})ds$$

$$\therefore\ \vec{v}\cdot d\vec{l}\Big|_{\text{curve}} = 2\cdot(8s^4+8s^5) + 2s(8s^3+16s^4) + 2(4s^4+24s^5)$$

$$= 16s^4+16s^5+16s^4+32s^5+8s^4+48s^5$$

$$= 40s^4+96s^5$$

to get from $\vec{a}=0$ to $\vec{b}=2\hat{x}+\hat{y}+2\hat{z}$, $s\in[0,1]$, $\rightarrow$

$$I = \int_a^b \vec{v}\cdot d\vec{l} = \int_0^1 (40s^4+96s^5)ds = \frac{40}{5}+\frac{96}{6} = 8+16 = 24$$

Problem 5

$T = x^2yz + xyz^3$ has $\nabla T = \vec{v}$.

$T(\vec{b}) - T(\vec{a}) = 2^2\cdot1\cdot2 + 2\cdot1\cdot2^3 = 8+16 = 24$ ✓

Problem 6

For $\vec{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$ we want to show that:

$$\int_\Omega \nabla\cdot\vec{v}\,d\tau = \oint_{\partial\Omega}\vec{v}\cdot d\vec{a}$$ for a cube of side length 2.

$\nabla\cdot\vec{v} = y+2z+3x$, $\therefore$

$$\int_\Omega \nabla\cdot\vec{v}\,d\tau = \int_0^2\int_0^2\int_0^2 (y+2z+3x)\,dx\,dy\,dz = \int_0^2\int_0^2(2y+4z+6)\,dy\,dz$$

$$= \int_0^2 (4+8z+12)\,dz = 8+16+24 = 48$$

$$\int_{\partial\Omega}\vec{v}\cdot d\vec{a} = \int_0^2\int_0^2 \vec{v}\Big|_{x=2}\cdot dy\,dz\,\hat{x} + \int_0^2\int_0^2 \vec{v}\Big|_{y=2}\cdot dx\,dz\,\hat{y} + \int_0^2\int_0^2 \vec{v}\Big|_{z=2}\cdot dx\,dy\,\hat{z}$$

$$- \int_0^2\int_0^2 \vec{v}\Big|_{x=0}\cdot dy\,dz\,\hat{x} - \int_0^2\int_0^2 \vec{v}\Big|_{y=0}\cdot dx\,dz\,\hat{y} - \int_0^2\int_0^2 \vec{v}\Big|_{z=0}\cdot dx\,dy\,\hat{z}$$

$$= \int_0^2\int_0^2 2y\,dy\,dz + \int_0^2\int_0^2 4z\,dx\,dz + \int_0^2\int_0^2 6x\,dx\,dy$$

$$= \int_0^2 4\,dz + \int_0^2 8\,dx + \int_0^2 12\,dy = 8+16+24 = 48$$ ✓

Problem 7 (1.61)

a. $$\int_\Omega \nabla\cdot(\vec{c}T)\,d\tau = \oint_{\partial\Omega} \vec{c}T\cdot d\vec{a}$$ from the div. thm.

$\parallel$ (product rule) $\quad \parallel$

$$\vec{c}\cdot\int_\Omega \nabla T\,d\tau = \vec{c}\cdot\oint T\,d\vec{a}$$ true $\forall\ \vec{c}$, so $$\int_\Omega \nabla T\,d\tau = \oint_{\partial\Omega} T\,d\vec{a}$$

b. $$\int_\Omega \nabla\cdot(\vec{v}\times\vec{c})\,d\tau = \oint_{\partial\Omega}(\vec{v}\times\vec{c})\cdot d\vec{a}$$ from the div. thm

$\parallel$ product rule $\quad \parallel\ d\vec{a}\cdot(\vec{v}\times\vec{c}) = \vec{c}\cdot(d\vec{a}\times\vec{v}) = -\vec{c}\cdot(\vec{v}\times d\vec{a})$

$$\vec{c}\cdot\int_\Omega(\nabla\times\vec{v})\,d\tau = -\vec{c}\cdot\oint_{\partial\Omega}\vec{v}\times d\vec{a}$$ true $\forall\ \vec{c}$, so $$\int_\Omega \nabla\times\vec{v}\,d\tau = -\oint_{\partial\Omega}\vec{v}\times d\vec{a}.$$

c. $$\int_\Omega \nabla\cdot(T\nabla u)\,d\tau = \oint_{\partial\Omega}(T\nabla u)\cdot d\vec{a}$$ from the div. thm.

$\parallel$ product rule.

$$\int_\Omega\left[T\nabla^2 u + \nabla T\cdot\nabla u\right]d\tau = \oint_{\partial\Omega}(T\nabla u)\cdot d\vec{a}$$

d. from part c. $$\int_\Omega\left[T\nabla^2 u + \nabla T\cdot\nabla u\right]d\tau = \oint_{\partial\Omega}(T\nabla u)\cdot d\vec{a}$$

interchange $T\leftrightarrow u$: $$\int_\Omega\left[u\nabla^2 T + \nabla u\cdot\nabla T\right]d\tau = \oint_{\partial\Omega}(u\nabla T)\cdot d\vec{a}$$

subtract to get $$\int_\Omega\left[T\nabla^2 u - u\nabla^2 T\right]d\tau = \oint\left[T\nabla u - u\nabla T\right]\cdot d\vec{a}$$

e. $$\int_S\left[\nabla\times(\vec{c}T)\right]\cdot d\vec{a} = \oint_{\partial S} T\vec{c}\cdot d\vec{l}$$ from Stokes's theorem

$\parallel$ product rule

$$\int_S\left[T\cancel{\nabla\times\vec{c}} - \vec{c}\times\nabla T\right]\cdot d\vec{a} = \vec{c}\cdot\oint_{\partial S} T\,d\vec{l} \Rightarrow \int_S \underbrace{d\vec{a}\cdot(\vec{c}\times\nabla T)}_{=\vec{c}\cdot(\nabla T\times d\vec{a})} = -\vec{c}\cdot\int_{\partial S} T\,d\vec{l}$$

so $$\vec{c}\cdot\int_S(\nabla T\times d\vec{a}) = -\vec{c}\cdot\int_{\partial S} T\,d\vec{l},$$ true $\forall\ \vec{c}$, so $$\int_S \nabla T\times d\vec{a} = -\int_{\partial S} T\,d\vec{l}$$

Problem 8 | | Problem Set 2 (continued)

For $r = \sqrt{x^2+y^2+z^2}$, we have $\frac{\partial r}{\partial x} = \frac{\frac{1}{2}\cdot 2x}{r} = \frac{x}{r}$, & sim. $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

so $\nabla r = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{1/2}} = \frac{\vec{r}}{r} = \hat{r}$ ✓

Problem 9

let $h(y) = \frac{dg(y)}{dy}$, then $\frac{d}{dx}\left[\int_0^x h(y)\,dy\right] = \frac{d}{dx}\left[\int_0^x \frac{dg(y)}{dy}\,dy\right]$

$= \frac{d}{dx}\left[g(x) - g(0)\right]$

$= \frac{dg(x)}{dx} = h(x)$

so $\frac{d}{dx}\int_0^x h(y)\,dy = h(x)$