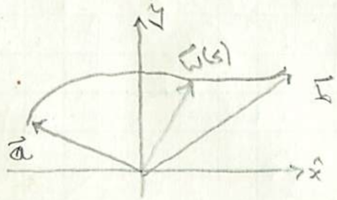
Line Integrals

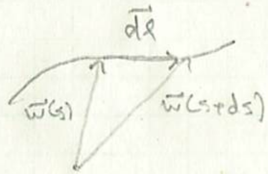
A curve can be described by a vector $\vec{w}(s)$:



w/ $\vec{w}(s_0) = \vec{a}$
$\vec{w}(s_f) = \vec{b}$

for parameter $s \in [s_0, s_f]$.

The tangent to the curve, $d\vec{l}$, is related to the derivative of $\vec{w}(s)$:



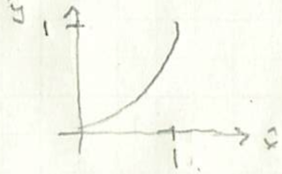
$$d\vec{l} = \lim_{ds \to 0} [\vec{w}(s+ds) - \vec{w}(s)]$$
$$= \frac{d\vec{w}(s)}{ds} ds$$

A typical type of integral along a curve has the form:

$$I = \int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{l} = \int_{s_0}^{s_f} \vec{V} \cdot \frac{d\vec{w}}{ds} ds$$

for a vector function $\vec{V}$ (think of work)

<u>example</u>: For the curve $\vec{w}(s) = s\,\hat{x} + s^2\,\hat{y}$ w/ $s \in [0,1]$



$$\frac{d\vec{w}}{ds} = \hat{x} + 2s\,\hat{y}$$

along the curve, we have: $x = s$, $y = s^2$

take $\vec{V} = \alpha x\,\hat{x} + \beta x y\,\hat{y}$ as our vector function evaluated along the curve,

$$\vec{V}\Big|_{\vec{w}} = \alpha \underbrace{s}_{x}\,\hat{x} + \beta \underbrace{s}_{x}\,\underbrace{s^2}_{y}\,\hat{y}$$

+

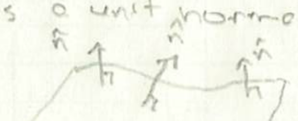
$$\vec{V}\Big|_{\vec{w}} \cdot d\vec{l} = (\alpha s\,\hat{x} + \beta s^3\,\hat{y}) \cdot (\hat{x} + 2s\,\hat{y})\, ds$$
$$= (\alpha s + 2\beta s^4)\, ds$$

+

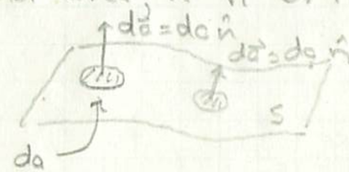
$$I = \int_{\vec{a}=0}^{\vec{b}=\hat{x}+\hat{y}} \vec{V} \cdot d\vec{l} = \int_0^1 (\alpha s + 2\beta s^4)\, ds = \tfrac{1}{2}\alpha + \tfrac{2}{5}\beta$$

Surface Integrals

A surface has a unit normal (that can vary over the surface)



define $d\vec{a}$ to be a vector w/ magnitude that is an infinitesimal area, + $\hat{n}$ as it direction

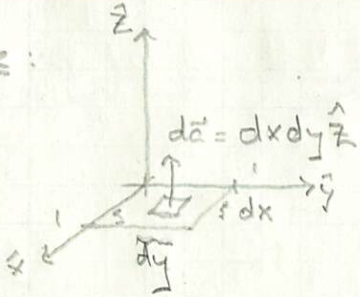


Given a vector function $\vec{V}$, we are interested in

$$I = \int_S \vec{V} \cdot d\vec{a}$$

(flux, for instance, hence the dot product).

<u>example</u>:



(surface has $z=0$)

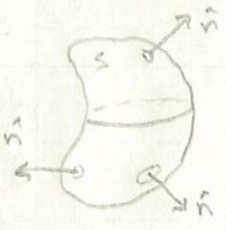
take $\vec{V} = \alpha x y\,\hat{z}$, then

$$\int_S \vec{V}\cdot d\vec{a} = \int_0^1\int_0^1 (\alpha x y\,\hat{z})\cdot(dx\,dy\,\hat{z})$$

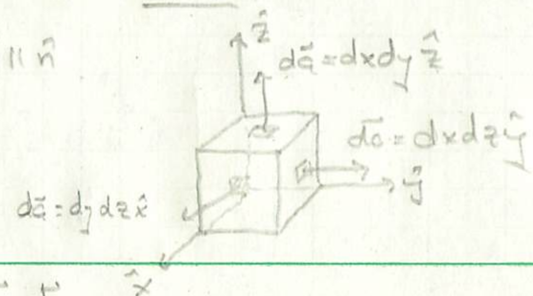
$$= \int_0^1\int_0^1 \alpha x y\,dx\,dy = \int_0^1 \frac{1}{2}\alpha y\,dy = \frac{1}{4}\alpha$$

"open surfaces" have $\hat{n}$ that can be "above" or "below"

"closed surfaces," for which their is an inside & an outside have unit <u>outward</u> normal



w/ $d\vec{a} \parallel \hat{n}$



integration over a closed surface: $\oint \vec{V}\cdot d\vec{a}$

<u>Volume Integrals</u>

Given a volume Ω we have infinitesimal volume elt $d\tau$

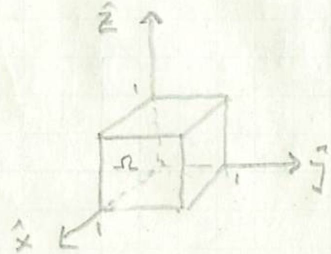
w/

$d\tau = dx\,dy\,dz$ (in Cartesian coordinates)

here, the main integral of interest is

$$I = \int_\Omega f\,d\tau \quad \text{for a function } f(x,y,z)$$

<u>example:</u>



take

$f = \alpha x^2 y z$

$$\int_\Omega f\,d\tau = \int_0^1\int_0^1\int_0^1 \alpha x^2 y z\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \frac{1}{3}\alpha y z\,dy\,dz = \int_0^1 \frac{1}{6}\alpha z\,dz = \frac{1}{12}\alpha$$

<u>Fundamental Theorem(s) of Calculus</u>

In one dimension: $I = \int_a^b \frac{df(x)}{dx}dx = f(b) - f(a)$

the integral of the derivative of $f(x)$ for $x\in[a,b]$ is equal to f evaluated on the boundaries of the domain.

there are different flavors for line, surface, & volume integrals,

1. line integrals: $I = \int_{\vec a}^{\vec b} \vec v \cdot d\vec l$

we want $\vec v$ to "be a derivative" but of what?

one option: $\vec v = \nabla f$

(aside – notation $f(x,y,z) \to f(\vec r)$ w/ $\vec r = x\hat x + y\hat y + z\hat z$)

For a curve specified by $\vec w(s)$, w/ $d\vec l = \frac{d\vec w(s)}{ds} ds$, we have

$$\int_{\vec a}^{\vec b} \nabla f\cdot d\vec l = \int_{s_0}^{s_e}\nabla f\cdot d\vec l \quad \text{(eval. along the curve)} \quad \& \quad \frac{df(\vec w(s))}{ds} = \nabla f\cdot \frac{d\vec w}{ds}$$

$$= \int_{s_0}^{s_e} \frac{df}{ds}\, ds = f(\underbrace{\vec w(s_e)}_{=\vec b}) - f(\underbrace{\vec w(s_0)}_{=\vec a})$$

$$= f(\vec b) - f(\vec a)$$

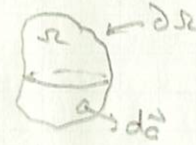
The result,

$$\int_{\vec a}^{\vec b}\nabla f\cdot d\vec l = f(\vec b)-f(\vec a) \quad (*)$$

is familiar from conservative forces ($\vec F = -\nabla U$) when calculating work.

Note that $\oint \nabla f\cdot d\vec l = 0$ follows from (*)

2. Volume integrals:



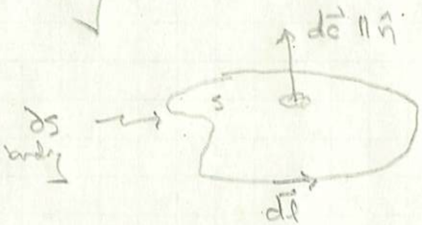
$$\int_\Omega (\nabla\cdot\vec v)\, d\tau = \oint_{\partial\Omega} \; ?$$

derivative of $\vec v$ evaluated "in the bulk" — $\vec v$ evaluated on surface

the "?" must be a scalar – most natural is $\vec v\cdot d\vec a$ (gets the D=1 limit right)

$$\int_\Omega(\nabla\cdot\vec v)\, d\tau = \oint_{\partial\Omega}\vec v\cdot d\vec a \quad \text{(divergence theorem)}$$

3. Surface integrals: for open S w/ bndry ∂S



choice of $\vec n$ determines direction of $d\vec l$ via right-hand rule.

$$\int_S (\nabla\times\vec v)\cdot d\vec a = \int_{\partial S}(\text{scalar} \sim \vec v\text{, must involve } d\vec l) \; \ldots$$