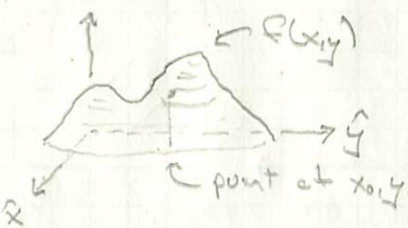
Gradient



height map.

Given $f(x,y)$ - at a point $\vec{r}_0 = x_0\hat{x} + y_0\hat{y}$

the "height" is $f(x_0, y_0)$ - how should we pick dx_0 + dy_0 in

$$\vec{dl} = dx_0\hat{x} + dy_0\hat{y}$$

such that $f(x_0+dx_0, y_0+dy_0) \geq f(x_0,y_0)$?

$$f(x_0+dx_0, y_0+dy_0) \simeq f(x_0,y_0) + dx_0 \frac{\partial f}{\partial x}\bigg|_{x_0, y_0} + dy_0 \frac{\partial f}{\partial y}\bigg|_{x_0, y_0} \quad (*)$$

Define the "gradient" of $f(x,y)$:

$$\nabla f \equiv \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

then $(*)$ can be written:

$$f(x_0+dx_0, y_0+dy_0) \simeq f(x_0,y_0) + \underbrace{\nabla f \cdot \vec{dl}}_{\text{want } \geq 0}$$

we can maximize the dir. of increase:

$$\nabla f \cdot \vec{dl} = \|\nabla f\| dl \cos\theta \leftarrow \text{angle between } \nabla f \text{ + } \vec{dl}$$

by taking $\vec{dl} \parallel \nabla f$. $(\theta = 0)$

in $D=3$: $\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ $\quad (+)$

is a vector operator. The gradient of $f(x,y,z)$,

∇f gives direction of greatest increase of f.

Divergence + Curl

Taking $(+)$ "seriously" as a vector, we can define the divergence of a vector function

$$\vec{A} = A^x(x,y,z)\hat{x} + A^y(x,y,z)\hat{y} + A^z(x,y,z)\hat{z}$$

via

$$\nabla \cdot \vec{A} = \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z},$$

and the "curl"

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A^x & A^y & A^z \end{vmatrix} = \hat{x}\left(\frac{\partial A^z}{\partial y} - \frac{\partial A^y}{\partial z}\right) - \hat{y}\left(\frac{\partial A^z}{\partial x} - \frac{\partial A^x}{\partial z}\right)$$

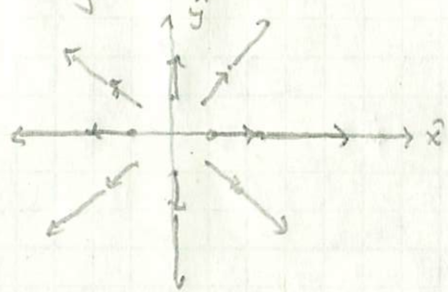
$$+ \hat{z}\left(\frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y}\right)$$

Some examples help w/ the names - define the vector functions:

$$\vec{U} \equiv x\hat{x} + y\hat{y} \quad + \quad \vec{V} \equiv -y\hat{x} + x\hat{y}$$

we can sketch vector functions by drawing vectors at a sampling of points:

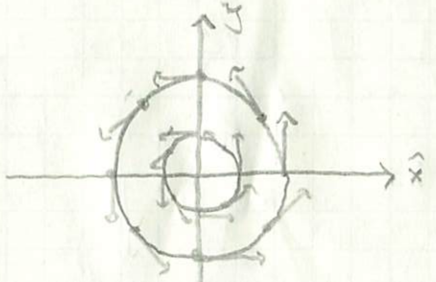
sketch of $\vec{u}$:



The divergence of $\vec{u}$ is: $\nabla\cdot\vec{u} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} = 2$

The curl of $\vec{u}$ is: $\nabla\times\vec{u} = \hat{z}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$

sketch of $\vec{v}$:



The divergence of $\vec{v}$ is: $\nabla\cdot\vec{v} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} = 0$

w/ curl $\nabla\times\vec{v} = \hat{z}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) = 2\hat{z}$

dir. given by right-hand rule for curls

Example

Take $\vec{A} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$? What does a sketch of $\vec{A}$ look like?

(points radially away from the origin, gets smaller in magnitude as you go further from the origin).

What is the divergence of $\vec{A}$?

let's compute the "first" term of the sum:

$$\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3/2\,x\cdot 2x}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}$$

then, w/ similar expressions for the y & z derivs of the y & z components, we have:

$$\nabla\cdot\vec{A} = \frac{[-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2]}{(x^2+y^2+z^2)^{5/2}} = 0$$

... ?!

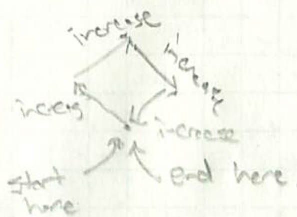
Second Derivatives

A lot of options here:

$\nabla\cdot(\nabla f)$, $\nabla\cdot(\nabla\times\vec{A})$, $\nabla\times(\nabla f)$, $\nabla\times(\nabla\times\vec{A})$, $\nabla(\nabla\cdot\vec{A})$

Consider $\nabla\times(\nabla f)$ first: ∇f points in the direction of increasing f.

suppose $\nabla\times(\nabla f)\neq 0$ → ∇f "curls around"



this cannot happen, so

$\nabla\times(\nabla f)=0$.

"similarly" $\nabla\cdot(\nabla\times\vec{A})=0$

example: $\vec{A}=A^x(x,y)\hat{x}+A^y(x,y)\hat{y}$

$$\nabla\times\vec{A}=\left(\frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y}\right)\hat{z} \to \nabla\cdot(\nabla\times\vec{A})=\frac{\partial}{\partial z}\left(\frac{\partial A^y}{\partial x}-\frac{\partial A^x}{\partial y}\right)=0$$

The only non-identically-zero combos are:

$$\nabla\cdot(\nabla f)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\underbrace{\nabla^2 f}_{\text{"Laplacian"}}, \quad \nabla(\nabla\cdot\vec{A}),$$

$$\to \nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

note that

$$\nabla^2\vec{A}=\nabla^2(A^x\hat{x})+\nabla^2(A^y\hat{y})+\nabla^2(A^z\hat{z})$$

$$=(\nabla^2A^x)\hat{x}+(\nabla^2A^y)\hat{y}+(\nabla^2A^z)\hat{z}$$

since $\nabla^2\hat{x}=\nabla^2\hat{y}=\nabla^2\hat{z}=0$

careful - this is not true for basis vectors that depend on position.