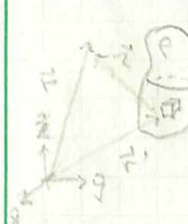
Last Time



$$\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} d\tau'$$

solving $\nabla\cdot\vec{E} = \rho/\epsilon_0$, $\nabla\times\vec{E}=0$

If we take $\vec{E} = -\nabla V$ then $\nabla\times\vec{E}=0$ automatically, &

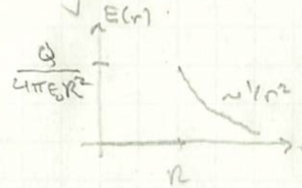
$\nabla^2 V = -\rho/\epsilon_0$ w/ soltn.

$$V(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}')}{4\pi\epsilon_0 \mathscr{r}} d\tau'$$

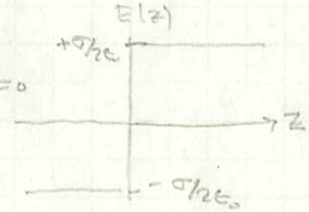
easier integral (& fewer!)

Properties of V & $\vec{E}$

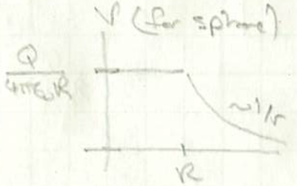
The electric field suffers a discontinuity at surface charges:

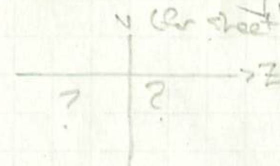


$$\sigma_0 = \frac{Q}{4\pi R^2}$$



the electrostatic potential V does not have discontinuities at surface charges

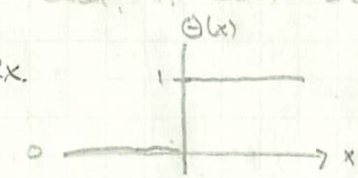


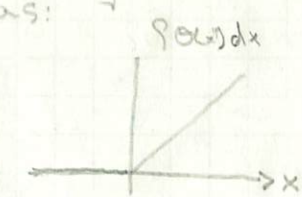


Since $\vec{E} = -\nabla V$, V is related to $\vec{E}$ by an integral $\left(V(\vec{r}) = V(\vec{a}) - \int_{\vec{a}}^{\vec{r}} \vec{E}\cdot d\vec{l}\right)$

The integral of a discontinuity, like a step function, is continuous:

ex.

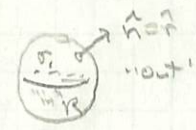




incidentally, that observation helps us understand what happens to $\vec{E}$ at a discontinuity in ρ... $\boxed{\rho_0}$ o

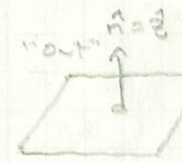
Discontinuity of $\vec{E}$ at a surface charge

We have 2 examples so far:



$$\vec{E}_{out} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \qquad \vec{E}_{in} = 0$$

$$\left(\vec{E}_{out} - \vec{E}_{in}\right)\Big|_{r=R} = \underbrace{\frac{Q}{4\pi R^2}}_{\sigma_0} \frac{1}{\epsilon_0}\hat{r} = \frac{\sigma_0}{\epsilon_0}\hat{r}$$



$$\vec{E}_{out} = \sigma_0/2\epsilon_0\,\hat{z} \qquad \vec{E}_{in} = -\sigma_0/2\epsilon_0\,\hat{z}$$

w/ $\left(\vec{E}_{out} - \vec{E}_{in}\right)\Big|_{z=0} = \frac{\sigma_0}{\epsilon_0}\hat{z}$

Let's check another example - find $\vec{E}$ for a cylinder:

1. "cylindrical symmetry" $\vec{E} = E(s)\hat{s}$

2. 

$$\oint_S \vec{E}\cdot d\vec{a} = E(s)\cdot 2\pi s h$$

$$Q_{enc} = \begin{cases} 0 & s<R \\ 2\pi R h\sigma_0 & s>R \end{cases}$$

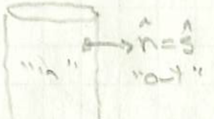
4. for $s<R$: $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$E(s)\cdot 2\pi s h = 0 \Rightarrow \vec{E}_{in} = 0$

for $s > R$, $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$$E(s)\cdot 2\pi s h = \frac{2\pi R h \sigma_0}{\epsilon_0}$$

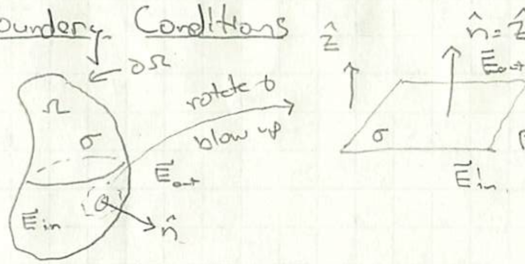
so $\vec{E}_{out} = \frac{\sigma_0 R}{\epsilon_0 s}\hat{s}$



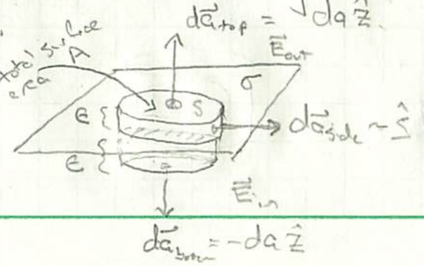
$$(\vec{E}_{out} - \vec{E}_{in})\big|_{s=R} = \frac{\sigma_0}{\epsilon_0}\hat{s}$$

It looks like in general... $(\vec{E}_{out} - \vec{E}_{in})\big|_{surf} = \sigma/\epsilon_0\,\hat{n}$

Boundary Conditions



apply Gauss's Law to a cylinder straddling this surface. $d\vec{a}_{top} = da\,\hat{z}$



Gauss's Law is $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

at the surface, $\vec{E}$ has components $\parallel$ & $\perp$ to the surface

$$\vec{E} = \underbrace{\vec{E}^{\parallel}}_{\hat{x},\hat{y}} + \underbrace{\vec{E}^{\perp}}_{\hat{z}}$$

let's compute the integral over the surface:

$$\oint_S \vec{E}\cdot d\vec{a} = \int_{top} \vec{E}_{out}\cdot d\vec{a}_{top} + \int_{bottom} \vec{E}_{in}\cdot d\vec{a}_{bottom} + \underbrace{\int_{side} \vec{E}\cdot d\vec{a}_{side}}_{\text{as } \epsilon\to 0 \text{, this term will vanish}}$$

$$\int_{top} \vec{E}_{out}\cdot d\vec{a}_{top} = \int_{top} \vec{E}^{\perp}_{top}\cdot da\,\hat{z} = E^{\perp}_{out}A$$

for small A

$$\int_{bottom} \vec{E}_{in}\cdot d\vec{a}_{bottom} = -\int_{bottom} \vec{E}^{\perp}\cdot d\vec{a}_{bottom} = -E^{\perp}_{in}A$$

The charge enclosed by the cylinder is $Q_{enc} = \sigma A$

so

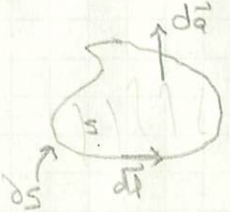
$$\lim_{\epsilon\to 0}\oint_S \vec{E}\cdot d\vec{a} = \frac{\sigma A}{\epsilon_0} \Rightarrow (E^{\perp}_{out} - E^{\perp}_{in})\big|_{surf.}\cdot A = \frac{\sigma A}{\epsilon_0}$$

or $(E^{\perp}_{out} - E^{\perp}_{in})\big|_{surf} = \sigma/\epsilon_0$ <u>discontinuous</u> $\perp$ component

What about the parallel components of $\vec{E}$ - here, the $\hat{x}$ & $\hat{y}$ pieces?

We can use $\nabla\times\vec{E}=0$ to find out what happens to the field in the $\parallel$ directions

Note that

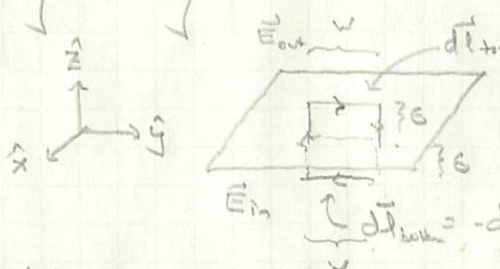


$$\int_S (\nabla\times\vec{E})\cdot d\vec{a} = 0$$

$\parallel$ Stoke's Thm $\parallel$

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = 0$$

Now for our zoom-in surface, orient a loop w/ legs of height ϵ



for this loop, we have:

$$\oint \vec{E}\cdot d\vec{l} = \int \vec{E}_{out}\cdot d\vec{l}_{top} + \int \vec{E}_{in}\cdot d\vec{l}_{bottom} + \underbrace{\int \vec{E}\cdot d\vec{l}_{sides}}_{\text{this term will go to zero as } \epsilon\to 0} = 0$$

for small w, we have

$$\lim_{\epsilon\to 0}\oint \vec{E}\cdot d\vec{l} = \left(E^y_{out}\,w - E^y_{in}\,w\right)\Big|_{surf} = 0$$

+ sim. for $\hat{x}$-oriented loop.

or

$$\left(\vec{E}^{\parallel}_{out} - \vec{E}^{\parallel}_{in}\right)\Big|_{surf} = 0$$

parallel components are continuous

Together: $\left(\vec{E}_{out}-\vec{E}_{in}\right)\big|_{surf} = \sigma/\epsilon_0\,\hat{n}$ ✓ (*)

Potential

The potential $V(\vec{r})$ is continuous everywhere, but has a discontinuous derivative — using $\vec{E} = -\nabla V$ in (*)

$$-\left(\nabla V_{out} - \nabla V_{in}\right)\Big|_{surf} = \frac{\sigma}{\epsilon_0}\hat{n}$$

dot $\hat{n}$ into both sides:

$$\left(\frac{\partial V_{out}}{\partial n} - \frac{\partial V_{in}}{\partial n}\right)\Big|_{surf} = -\frac{\sigma}{\epsilon_0}$$

w/ $\left.\frac{\partial V}{\partial n}\right|_{surf} = \nabla V\cdot\hat{n}\Big|_{surf}$