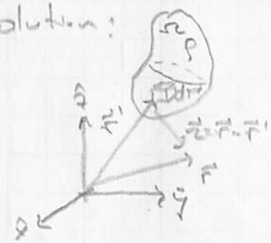
The Other Equation

We took the integral solution:

$$\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')\hat{\mathscr{r}}\, d\tau'}{4\pi\epsilon_0 \mathscr{r}^2}$$



& found:

$$\nabla\cdot\vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0 \quad \& \quad \nabla\times\vec{E}(\vec{r}) = 0$$

Gauss's Law ↑ ↑ ?

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0}$$

The divergence relates the source (ρ) to the field.

What does $\nabla\times\vec{E}(\vec{r}) = 0$ tell us?

? How can we get $\nabla\times\vec{E} = 0$?

ex: $\vec{E} = E_0\hat{x}$ has $\nabla\times\vec{E} = 0$

& even $\vec{E} = \alpha x\hat{x}$ has $\nabla\times\vec{E} = 0$

what's the most general way to ensure that $\nabla\times\vec{E} = 0$ ("electric field is curl-free") ?

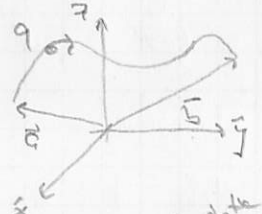
Recall the identity, for any $f(\vec{r})$:

$$\nabla\times[\nabla f] = 0$$

(can't always go uphill & end at your starting point).

so setting: $\vec{E}(\vec{r}) = \overset{\text{convention}}{-}\nabla V(\vec{r})$ is the most general way to ensure that $\nabla\times\vec{E}(\vec{r}) = 0$.

The work done by an electric force as a particle moves from $\vec{a}\rightarrow\vec{b}$ is:



$$W = \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l} = q\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$$

$$= -q\int_{\vec{a}}^{\vec{b}} \nabla V\cdot d\vec{l} = -q[V(\vec{b}) - V(\vec{a})]$$

"gradient thm" (one of the fund. thms. of Calc)

"so" $V(\vec{r})$ ("electrostatic potential") has units of J/C, the work done is path-indep. (depends only on the end points), & the work done around a closed loop, w/ $\vec{b}=\vec{a}$, is:

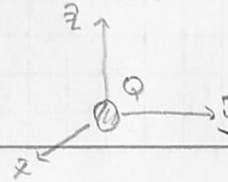
$$W = -q[V(\vec{b}=\vec{a}) - V(\vec{a})] = 0 \quad \text{(no work done)}$$

Given $\vec{E}$, we can integrate along any path from $\vec{a}$ to $\vec{r}$ to get $V(\vec{r})$:

$$\int_{\vec{a}}^{\vec{r}} \vec{E}\cdot d\vec{l} = -\int_{\vec{a}}^{\vec{r}} \nabla V\cdot d\vec{l} = -[V(\vec{r}) - V(\vec{a})] \Rightarrow V(\vec{r}) = V(\vec{a}) - \int_{\vec{a}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

the more common usage direction for V is to get $\vec{E}$ from V via $\vec{E} = -\nabla V$.

In order to do that, we need V associated w/ a pt. charge



$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ w/ $\vec{E} = -\nabla V$ - take $V(\vec{r}) = V(r)$, then

$$\vec{E} = -\nabla V = -\frac{dV}{dr}\hat{r} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \Longrightarrow V = \frac{Q}{4\pi\epsilon_0 r}.$$

integrating

A point charge at the origin has potential

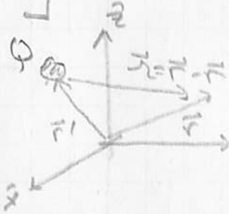
$V(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r}$ note that $\bar{V}(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 r} + \underbrace{V_0}_{\text{constant}}$

has

$\vec{E} = -\nabla V = -\nabla \bar{V}$, so that V & $\bar{V}$ produce the same electric field.

There is a "boundary condition" here – we set $V(r \to \infty) = 0$ then $V_0 = 0$; this is our choice.

Moving the source charge Q to location $\vec{r}\,'$:



this is a shift of origin, so $r \to \mathscr{r}$ (from $\vec{r} \to \vec{\mathscr{r}}$) & $V(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0 \mathscr{r}}$.

(Check: $\vec{E}(\vec{r}) = -\nabla V(\vec{r}) = \dfrac{Q\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$ ✓)

Superposition

Superposition holds for V just as it does for $\vec{E}$:

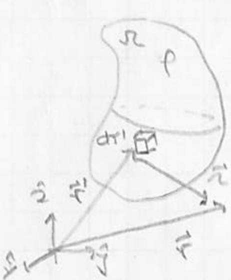
$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots$

w/ $\vec{E}_1 = -\nabla V_1$, $\vec{E}_2 = -\nabla V_2$, $\cdots$ gives

$-\nabla V = -\nabla V_1 - \nabla V_2 - \cdots$

so $V = V_1 + V_2 + \cdots$

Consider a continuous volume charge distribution ρ,



$dQ = \rho(\vec{r}\,')\,d\tau'$ in the vol. $d\tau'$ at $\vec{r}\,'$

$dV = \dfrac{dQ}{4\pi\epsilon_0 \mathscr{r}}$ w/ $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ as usual.

$= \dfrac{\rho(\vec{r}\,')}{4\pi\epsilon_0 \mathscr{r}}\,d\tau'$

then, using superposition, we have

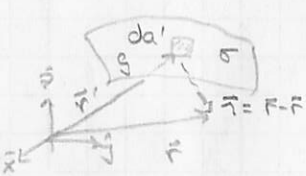
$$V(\vec{r}) = \int dV = \int_{\text{all space}} \frac{\rho(\vec{r}\,')\,d\tau'}{4\pi\epsilon_0 \mathscr{r}} \qquad (*)$$

this expression for $V(\vec{r})$ is 1/3 of the work required to calculate

$$\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')\,\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\,d\tau'$$

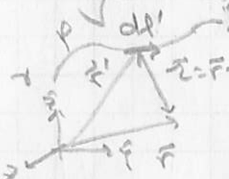
(we are, once again, solving $\nabla\cdot\vec{E} = \rho/\epsilon_0$, $\nabla\times\vec{E} = 0$ using $(*)$ w/ $\vec{E} = -\nabla V$, just a different approach.)

we have the usual degenerate charge distributions:



$dQ = \sigma(\vec{r}\,')\,da'$

$V(\vec{r}) = \displaystyle\int_S \frac{\sigma(\vec{r}\,')\,da'}{4\pi\epsilon_0 \mathscr{r}}$ &

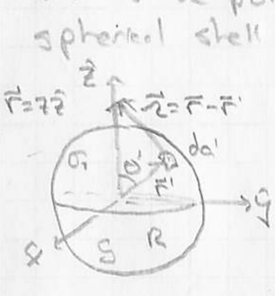


$dQ = \lambda(\vec{r}\,')\,dl'$

$V(\vec{r}) = \displaystyle\int_P \frac{\lambda(\vec{r}\,')\,dl'}{4\pi\epsilon_0 \mathscr{r}}$

Example - Spherical Shell

What is the potential inside & outside of a spherical shell of radius R w/ $\sigma_0 = \dfrac{Q}{4\pi R^2}$?



Take the "field point" $\vec{r}$ to be along the $\hat{z}$ axis:

$$\vec{r} = z\hat{z}$$

the source point, $\vec{r}'$, is at

$$\vec{r}' = R\sin\theta'\cos\phi'\,\hat{x} + R\sin\theta'\sin\phi'\,\hat{y} + R\cos\theta'\,\hat{z}$$

Then: $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}' = -R\sin\theta'\cos\phi'\,\hat{x} - R\sin\theta'\sin\phi'\,\hat{y} + (z - R\cos\theta')\,\hat{z}$

&

$$\mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = (R^2 + z^2 - 2zR\cos\theta')^{1/2}$$

so

$$V(\vec{r}) = \int_S \frac{\sigma(\vec{r}')}{4\pi\epsilon_0 \mathscr{r}}\,da' \quad \text{w/ } da' = R^2\sin\theta'\,d\theta'\,d\phi'$$

$$= \frac{\sigma_0}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{R^2\sin\theta'\,d\theta'\,d\phi'}{(R^2+z^2-2zR\cos\theta')^{1/2}}$$

$$= \frac{\sigma_0}{2\epsilon_0}R^2\int_0^{\pi}\frac{\sin\theta\,d\theta'}{(R^2+z^2-2zR\cos\theta')^{1/2}} \quad (+)$$

& note that (similar to previous integrands):

$$\frac{d}{d\theta'}\left(R^2+z^2-2zR\cos\theta'\right)^{+1/2} = \frac{+\frac{1}{2}\cdot 2zR\sin\theta'}{(R^2+z^2-2zR\cos\theta')^{1/2}}$$

gives:

$$\int\frac{\sin\theta'}{(R^2+z^2-2zR\cos\theta')^{1/2}} = +\frac{1}{zR}\left(R^2+z^2-2zR\cos\theta'\right)^{1/2}$$

so that (+) becomes:

$$V(\vec{r}) = \frac{\sigma_0}{2\epsilon_0}\cdot\frac{R^2}{zR}\left(R^2+z^2-2zR\cos\theta'\right)^{1/2}\Big|_{\theta'=0}^{\pi}$$

$$= \frac{\sigma_0 R}{2\epsilon_0 z}\Big[\underbrace{\sqrt{(R+z)^2}}_{=R+z} - \underbrace{\sqrt{(R-z)^2}}_{\substack{=R-z \quad z<R \\ z-R \quad z>R}}\Big]$$

for $z > R$, outside the shell, we have:

$$V_{out} = \frac{\sigma_0 R}{2\epsilon_0 z}\Big[R + z - (z - R)\Big] = \frac{\sigma_0 R^2}{\epsilon_0 z} = \frac{Q}{4\pi\epsilon_0 z} \xrightarrow{\text{general}} \frac{Q}{4\pi\epsilon_0 r}$$

& for $z < R$, inside the shell,

$$V_{in} = \frac{\sigma_0 R}{2\epsilon_0 z}\Big[R + z - (R - z)\Big] = \frac{\sigma_0 R}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R}$$

& $V_{out} = \dfrac{Q}{4\pi\epsilon_0 r}$ gives $\vec{E}_{out} = -\nabla V_{out} = \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}$ ✓

$V_{in} = \dfrac{Q}{4\pi\epsilon_0 R}$ has $\vec{E}_{in} = -\nabla V_{in} = 0$ ✓