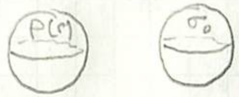
Using Gauss's Law to find $\vec{E}$

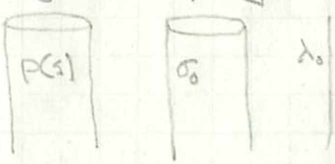
This can only be done in situations of source "symmetry" that implies a simplified form for the electric field:

- spherically symmetric sources



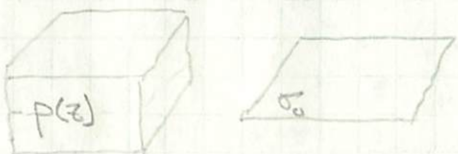
$$\vec{E} = E(r)\,\hat{r}$$

- cylindrically symmetric sources



$$\vec{E} = E(s)\,\hat{s}$$

- Cartesianly symmetric sources



$$\vec{E} = E(z)\,\hat{z}$$

once you have identified the correct source-field form pair, applying Gauss's Law:

$$\oint_{\partial\Omega} \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

is a delightful -step process

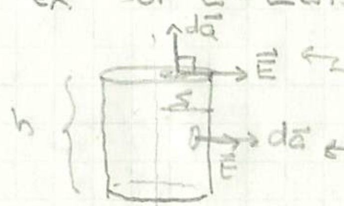
step 1 - determine if Gauss's Law is usefully applicable + write down the correct form of the field

ex: λ_0 we can use Gauss's Law; $\vec{E} = E(s)\,\hat{s}$

step 2 - pick a surface over which the evaluation of $\vec{E}\cdot d\vec{a}$ is "easy", meaning

1. $\vec{E} \parallel d\vec{a}$ w/ fixed argument
2. $\vec{E} \perp d\vec{a}$

ex for $\vec{E} = E(s)\hat{s}$, cylinders are "easy"-to-integrate surfaces



no contribution from top or bottom end "caps"

$$\vec{E}\cdot d\vec{a} = \underbrace{E(s)}_{s \text{ is fixed}}\,da = \underbrace{s\,d\phi\,dz}_{\text{no "}ds\text{"}}$$

on the curvy portion of the cylinder

step 3 - evaluate both sides of Gauss's Law for your surface separately

ex:

$$Q_{enc} = \lambda_0\cdot h \quad \Big(= \int_0^h \underbrace{\lambda_0\,\delta(x)\,\delta(y)}_{\rho}\,\underbrace{dx\,dy\,dz}_{d\tau}\Big)$$

$$\oint_{\partial\Omega}\vec{E}\cdot d\vec{a} = \overset{\text{top}}{0} + \overset{\text{bottom}}{0} + \int_{\text{side}}\vec{E}\cdot d\vec{a} = \int_{\text{side}} E(s)\,da = E(s)\underbrace{\int_{\text{side}} da}_{2\pi s h}$$

$$= E(s)\cdot 2\pi s h.$$

step 4 — put the 2 sides of Gauss's Law together to solve for the electric field magnitude:

$$\oint_{\partial\Omega} \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E(s)\cdot 2\pi s h = \frac{\lambda_0}{\epsilon_0} h \Rightarrow E(s) = \frac{\lambda_0}{2\pi\epsilon_0 s}$$

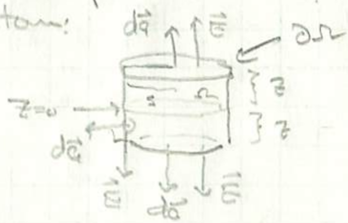
so we have $\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0 s}\hat{s}$, which we recognize as the correct field (we've computed it before).

let's do another example, this time w/ planar symmetry:

find $\vec{E}$ above & below an infinite sheet w/ uniform σ_0.

step 1 - planar symmetry w/ $\vec{E} = \begin{cases} E(z)\hat{z} & z>0 \\ -E(|z|)\hat{z} & z<0 \end{cases}$

step 2 - use a cylinder again - this time $\vec{E}\cdot d\vec{a} = 0$ for the curved portion, $\vec{E}\cdot d\vec{a} = E\,da$ for the top & bottom:



step 3 - $Q_{enc} = \sigma_0 \cdot \pi s^2$ ← area of disk in plane

$$\oint_{\partial\Omega} \vec{E}\cdot d\vec{a} = \int_{top} E(z)\,da + \int_{bottom} E(|z|)\,da + \int_{side} 0 \quad (\vec{E}\perp d\vec{a})$$

$$= E(z)\int_{top} da + E(|z|)\int_{bottom} da \quad \text{(same)}$$

$$= 2E(z)\cdot\pi s^2$$

step 4 - $\oint_{\partial\Omega} \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

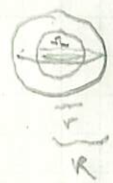
$$2E(z)\pi s^2 = \frac{1}{\epsilon_0}\sigma_0 \pi s^2 \Rightarrow E(z) = \frac{\sigma_0}{2\epsilon_0}$$

so $\vec{E} = \begin{cases} \sigma_0/2\epsilon_0\,\hat{z} & z>0 \\ -\sigma_0/2\epsilon_0\,\hat{z} & z<0 \end{cases}$

Example - Spherical Symmetry

let's do one where we have to be more careful about Q_{enc}

take a sphere w/ $\rho(r) = \alpha r$ - the total charge enclosed by a sphere of radius $r \le R$ is:
units?



$$Q(r) = \int_\Omega \rho(r)\,d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \alpha\bar{r}\,\bar{r}^2\sin\theta\,d\bar{r}\,d\theta\,d\phi$$

$$= 4\pi\alpha\int_0^r \bar{r}^3\,d\bar{r} = 4\pi\alpha\left(\tfrac{1}{4}r^4\right) = \alpha\pi r^4$$

the total charge enclosed by a sphere of radius $r > R$ is

$$Q(R) = \alpha\pi R^4$$

← no charge out here

all right — now we want to find $\vec{E}$ inside & outside of the charged sphere using Gauss's Law.

step 1 — spherical symmetry, $\vec{E} = E(r)\,\hat{r}$

step 2 — surface that has easy-to-evaluate $\vec{E}\cdot d\vec{a}$ is a sphere of radius r

$$\vec{E}\cdot d\vec{a} = E(r)\,da$$

← fixed

step 3 — we've already done the calculation:

$$Q_{enc.}(r) = \begin{cases} \alpha\pi r^4 & r \le R \\ \alpha\pi R^4 & r \ge R \end{cases}$$

& we can evaluate the surface integral:

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S E(r)\,da = E(r)\underbrace{\oint_S da}_{\text{surface area of a sphere of radius } r} = E(r)\cdot 4\pi r^2$$

step 4 — we have to put Gauss's Law together twice, one for $r < R$, one for $r > R$:

inside $\quad \oint_S \vec{E}\cdot d\vec{a} = \dfrac{Q_{enc.}}{\epsilon_0}$

$$E(r)\cdot 4\pi r^2 = \frac{\alpha}{\epsilon_0}\pi r^4 \Rightarrow E(r) = \frac{\alpha r^2}{4\epsilon_0}$$

so that $\quad \vec{E}_{in} = \dfrac{\alpha r^2}{4\epsilon_0}\,\hat{r} \quad$ (check using units of α...)

outside $\quad \oint \vec{E}\cdot d\vec{a} = \dfrac{Q_{enc.}}{\epsilon_0}$

$$E(r)\cdot 4\pi r^2 = \frac{\alpha}{\epsilon_0}\pi R^4 \Rightarrow E(r) = \frac{\alpha R^4}{4\epsilon_0 r^2}$$

so $\quad \vec{E}_{out} = \dfrac{\alpha R^4}{4\epsilon_0 r^2}\,\hat{r}$

the total charge of the sphere of radius R is: $Q_{tot} = Q(R) = \alpha\pi R^4$

then

$$\alpha = \frac{Q_{tot}}{\pi R^4} \quad \& \quad \vec{E}_{out} = \frac{Q_{tot}}{4\pi\epsilon_0 r^2}\,\hat{r} \quad ✓$$