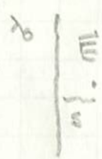
Last Time

We used

$$\vec{E} = \int_P \frac{\lambda(\vec{r}\,')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\,dl'$$

to find the field a distance s from an infinite wire w/ uniform λ



$$\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0 s}\hat{s}$$

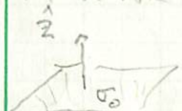
&

$$\vec{E} = \int_S \frac{\sigma(\vec{r}\,')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\,da'$$

to find $\vec{E}$ a height z above the center of a uniformly charged disk

$$\vec{E} = \frac{\sigma_0}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\right]\hat{z}$$

we can send $R \to \infty$ to get the field above an infinite sheet:



$$\vec{E} = \frac{\sigma_0}{2\epsilon_0}\hat{z}$$ (above the sheet)

Divergence & Curl of $\vec{E}$

From the general integral solution:

$$\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\,d\tau' \quad (*)$$

we can take the $\nabla\times$ & $\nabla\cdot$:

$$\nabla\times\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{4\pi\epsilon_0}\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

front cover of book

$$? \quad \nabla\times(f(\vec{r})\vec{A}(\vec{r})) = f\,\nabla\times\vec{A} - \vec{A}\times(\nabla f)$$

what happened to the: $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\times\left(\nabla\frac{\rho}{4\pi\epsilon_0}\right)$?

it will come as no surprise that

$$\nabla\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0$$

(but, by all means, check!).

How about the divergence?

$$\nabla\cdot\vec{E}(\vec{r}) = \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{4\pi\epsilon_0}\underbrace{\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{=4\pi\delta^3(\vec{r}-\vec{r}\,')}d\tau'$$

$$= \frac{\rho(\vec{r})}{\epsilon_0}$$ using the δ^3 to carry out the integration (sends $\vec{r}\,' \to \vec{r}$).

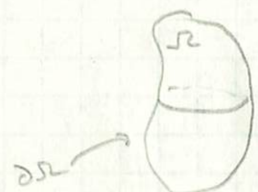
Knowing the divergence & curl of a vector field, together w/ relevant boundary conditions, gives a complete specification of the field – that's the content of Helmholtz's Theorem.

So $\nabla\times\vec{E} = 0$, $\nabla\cdot\vec{E} = \rho/\epsilon_0$ has basically the same content as $(*)$.

We'll come back to $\nabla\times\vec{E} = 0$ later on, for now, we'll focus on:

Gauss's Law

From $\nabla\cdot\vec{E} = \rho/\epsilon_0$, integrate both sides over a volume Ω:

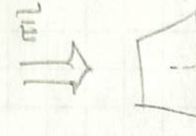


$$\underbrace{\int_\Omega \nabla\cdot\vec{E}\,d\tau}_{\text{" div. thm.}} = \frac{1}{\epsilon_0}\underbrace{\int_{\Omega}\rho\,d\tau}_{=Q_{enc.}\text{ the total charge in }\Omega}$$

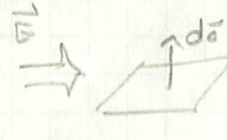
Gauss's Law (integral form)

$$\boxed{\oint_{\partial\Omega}\vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}}$$

On the left, we have a "flux integral" – tells us "how much" $\vec{E}$ flows through a surface:



vs.

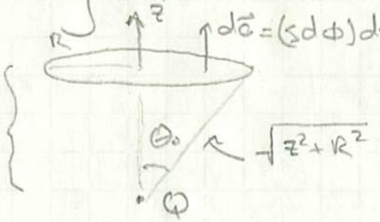


$\vec{E} \parallel d\vec{a}$ $\qquad$ $\vec{E}\cdot d\vec{a} = 0$

Example - Flux Integral

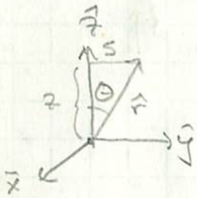
A charge Q sits at the origin, setting up the field $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$

? What is the flux of $\vec{E}$ through a disk of radius R a height z above the charge?



$\vec{E}\cdot d\vec{a} = \frac{Q}{4\pi\epsilon_0 r^2}\, s\,ds\,d\phi\,(\hat{r}\cdot\hat{z})$

w/ $r^2 = s^2 + z^2$ at the disk, &



$\hat{r}\cdot\hat{z} = \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{s^2+z^2}}$

so the integrand is

$$\vec{E}\cdot d\vec{a} = \frac{Q}{4\pi\epsilon_0 (s^2+z^2)}\underbrace{\frac{z}{\sqrt{s^2+z^2}}}_{\cos\theta}\, s\,d\phi\,ds$$

$$\& \quad \Phi \equiv \int \vec{E}\cdot d\vec{a} = \int_0^R\int_0^{2\pi} \frac{Qzs}{4\pi\epsilon_0 (s^2+z^2)^{3/2}}\,d\phi\,ds$$

$$= \frac{Qz}{2\epsilon_0}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds$$

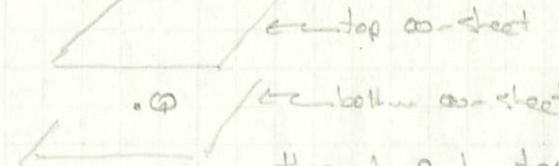
& we know, from last time, $\int \frac{s}{(s^2+z^2)^{3/2}}\,ds = -\frac{1}{\sqrt{s^2+z^2}}$

so

$$\Phi = -\frac{Qz}{2\epsilon_0}\frac{1}{\sqrt{s^2+z^2}}\Bigg|_{s=0}^{R} = \frac{Qz}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]$$

$$= \frac{Q}{2\epsilon_0}\left[1 - \cos\theta_0\right] \quad (*)$$

Check: what do we expect the flux to be if we send $R \to \infty$ ($\theta_0 \to \pi/2$)?



$\oint \vec{E}\cdot d\vec{a} = Q/\epsilon_0$

should get $\frac{1}{2}\frac{Q}{\epsilon_0}$ through each, top & bottom

From $(*)$: $\Phi = \frac{Q}{2\epsilon_0}\left[1 - \cos(\pi/2)\right] = \frac{Q}{2\epsilon_0}$ ✓

Gauss's Law - Utility

It's fun to compute the flux through surfaces & relate them to charge - but the more common application of Gauss's law is to find the electric field for sources w/ a "high degree of symmetry".

when the functional form of $\vec{E}(\vec{r})$ can be simplified

Here are the source configurations for which Gauss's law can be usefully applied to find $\vec{E}$:

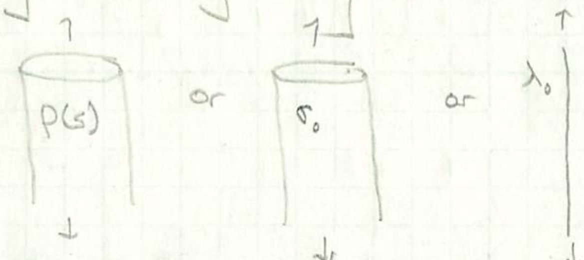
1. Spherical symmetry (of source):

(uniform surface charge density)

here, we start by arguing that $\vec{E} = E(r)\,\hat{r}$

(no preferred direction, & magnitude of $\vec{E}$ depends only on distance to origin).

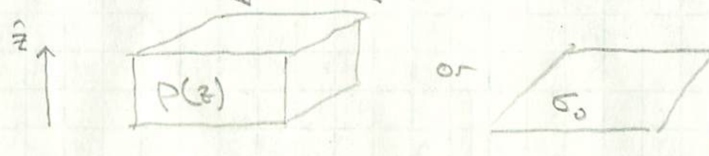
2. Cylindrical symmetry - for infinite distributions:



source is the same $\forall z$, & no distinction in ϕ, so

$$\vec{E} = \underbrace{E(s)\,\hat{s}}_{\text{indep. of } z \,\&\, \phi}$$ is the starting point.

3. Planar symmetry:



w/ the assumed form: $\vec{E} = E(z)\,\hat{z}$

(? why should the source "symmetry" influence the functional form of $\vec{E}$?)