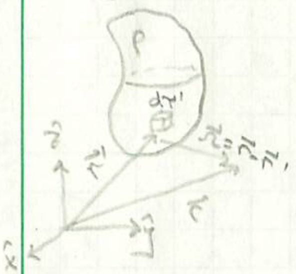
Integral Solutions for Electric Fields



In the small volume $d\tau'$ at $\vec{r}'$

$dq = \rho(\vec{r}')\, d\tau'$ ← volume charge distribution

& that contributes

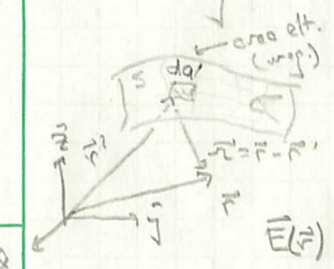
$d\vec{E} = \dfrac{dq\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}$ at the "field point" $\vec{r}$

$= \dfrac{\rho(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, d\tau'$

Sum up the contributions:

$$\vec{E}(\vec{r}) = \int d\vec{E} = \int_{\text{all space}} \frac{\rho(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, d\tau' = \int \frac{\rho(\vec{r}')\, \vec{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^3}\, d\tau'$$

We could also have a "surface charge" distribution, σ is charge/area

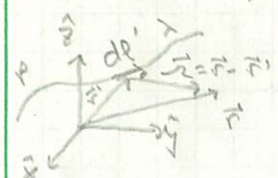


now $dq = \sigma(\vec{r}')\, da'$

$$d\vec{E} = \frac{dq\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} = \frac{\sigma(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, da'$$

$$\vec{E}(\vec{r}) = \int d\vec{E} = \int_S \frac{\sigma(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, da'$$

& a line charge distribution: λ is charge/len



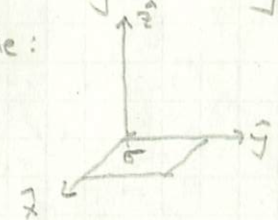
$dq = \lambda(\vec{r}')\, d\ell'$ w/

$$d\vec{E} = \frac{dq\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} = \frac{\lambda(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, d\ell'$$

$$\vec{E}(\vec{r}) = \int d\vec{E} = \int_P \frac{\lambda(\vec{r}')\, \hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}\, d\ell'$$

a surface charge density is a degenerate case of a volume charge density: $\rho \sim \sigma\,\delta$ ← Dirac delta distrib.

example:

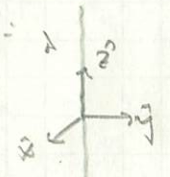


$\rho(x,y,z) = \sigma(x,y)\,\delta(z)$

(check units)

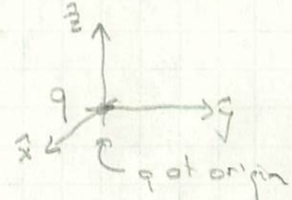
& a line charge density can also be expressed as a volume charge density: $\rho \sim \lambda\,\delta\,\delta$

example:



$\rho(x,y,z) = \lambda(z)\,\delta(x)\,\delta(y)$

even a point charge has a ρ:



$\rho(x,y,z) = q\,\delta^3(\vec{r})$

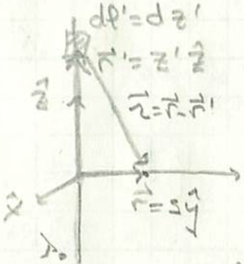
=

These integrals are solutions to the problem: "given ρ (or σ, λ) find $\vec{E}$"

But evaluating them can be difficult.

Example – Line of Uniform Charge Density

What is the electric field a distance s away from a line of charge w/ uniform λ_0?



take $\vec{r} = s\hat{y}$, we have

$$\vec{r}' = z'\hat{z} \ \& \ d\ell' = dz'$$

w/ $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = s\hat{y} - z'\hat{z}$, $\mathscr{r} = (s^2 + z'^2)^{1/2}$ all for use in our integral expression!

$$\vec{E} = \int_P \frac{\lambda(\vec{r}')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2} d\ell' = \int_P \frac{\lambda(\vec{r}')\vec{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^3} d\ell'$$

then:

$$\vec{E} = \frac{\lambda_0}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} \frac{s\hat{y} - z'\hat{z}}{(s^2 + z'^2)^{3/2}} dz' \qquad (+)$$

there are two separate integrals to know here, the first is

$$I_1 = \int \frac{z'}{(s^2 + z'^2)^{3/2}} dz' \ \& \text{ the second is } I_2 = \int \frac{1}{(s^2 + z'^2)^{3/2}} dz'$$

we'll see these over & over again, so let's figure them out once & for all!

I_1: we have z' in the numerator, z'^2 in the denom., smells like the chain rule in a deriv. Note that

$$\frac{d}{dz'}\left[\frac{\overset{\text{const.}}{\alpha}}{(s^2 + z'^2)^{1/2}}\right] = \frac{-\frac{1}{2}\alpha \cdot 2z'}{(s^2 + z'^2)^{3/2}} = \frac{-\alpha z'}{(s^2 + z'^2)^{3/2}}$$

& in order to match the integral in I_1, take $\alpha = -1$:

$$I_1 = \int \frac{z'}{(s^2 + z'^2)^{3/2}} dz' = -\frac{1}{(s^2 + z'^2)^{1/2}}$$

For I_2, note that

$$\frac{d}{dz'}\left[\frac{\overset{\text{const.}}{\alpha} z'}{(s^2 + z'^2)^{1/2}}\right] = \frac{\alpha}{(s^2 + z'^2)^{1/2}} - \frac{\frac{1}{2}\alpha \cdot 2z'}{(s^2 + z'^2)^{3/2}} \quad \text{make a common denominator}$$

$$= \frac{\alpha(s^2 + z'^2) - \alpha z'^2}{(s^2 + z'^2)^{3/2}} = \frac{\alpha s^2}{(s^2 + z'^2)^{3/2}}$$

& set $\alpha = 1/s^2$ (s is a constant w.r.t. z' deriv.s & int.s) to get the integrand in I_2:

$$I_2 = \int \frac{1}{(s^2 + z'^2)^{3/2}} dz' = \frac{z'/s^2}{(s^2 + z'^2)^{1/2}}.$$

Returning to (+), we have: how did we pull these out of the integrals?

$$\vec{E} = \frac{\lambda_0}{4\pi\epsilon_0}\left[\hat{y}\, s \int_{-\infty}^{+\infty} \frac{1}{(s^2 + z'^2)^{3/2}} dz' - \hat{z}\int_{-\infty}^{+\infty} \frac{z'}{(s^2 + z'^2)^{3/2}} dz'\right]$$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\left[\hat{y} \cdot s \left(\frac{1/s^2\, z'}{(s^2 + z'^2)^{1/2}}\right)\Bigg|_{z'=-\infty}^{+\infty} + \hat{z}\left(\frac{1}{(s^2 + z'^2)^{1/2}}\right)\Bigg|_{z'=-\infty}^{+\infty}\right]$$

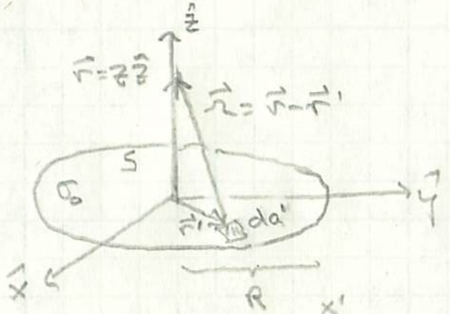
$: 0 = \frac{1}{\infty} - \frac{1}{\infty}$

$$= \frac{\lambda_0}{4\pi\epsilon_0}\hat{y}\frac{1}{s}\left(1 - (-1)\right) = \frac{\lambda_0}{2\pi\epsilon_0 s}\hat{y}$$

In general, for a distance s from the z axis, $\boxed{\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0 s}\hat{s}}$

Example - Uniform Disk of Charge

A disk of radius R carries uniform σ_0. It sits in the xy plane. What is $\vec{E}$ a height z above its center?



we have $\vec{r}=z\hat{z}$, $\vec{r}'=\underbrace{s'\cos\phi'\hat{x}}_{x'}+\underbrace{s'\sin\phi'\hat{y}}_{y'}$

+

$\vec{\mathscr{r}}=\vec{r}-\vec{r}'=-s'\cos\phi'\hat{x}-s'\sin\phi'\hat{y}+z\hat{z}$

w/

$\mathscr{r}=(s'^2+z^2)^{1/2}$

$da'=(s'd\phi')\cdot ds'$

We'll use these in the integral solution:

$$\vec{E}=\int_S \frac{\sigma(\vec{r}')\hat{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^2}da'=\int_S\frac{\sigma(\vec{r}')\vec{\mathscr{r}}}{4\pi\epsilon_0\mathscr{r}^3}da'$$

or, using our expressions above:

$$\vec{E}=\int_0^R\int_0^{2\pi}\frac{\sigma_0(-s'\cos\phi'\hat{x}-s'\sin\phi'\hat{y}+z\hat{z})}{4\pi\epsilon_0(s'^2+z^2)^{3/2}}\cdot s'd\phi'ds'$$

ϕ' integrals give zero for the $\hat{x}$ + $\hat{y}$ components, since $\int_0^{2\pi}\cos\phi'd\phi'=0=\int_0^{2\pi}\sin\phi'd\phi'$

+ we pick up a 2π for the $\hat{z}$ component (which does not depend on ϕ'):

$$\vec{E}=\frac{\sigma_0}{2\epsilon_0}z\hat{z}\int_0^R\frac{s'}{(s'^2+z^2)^{3/2}}ds'$$

the integral is like I_1 (w/ s' the integration variable now)

$$=\frac{\sigma_0 z}{2\epsilon_0}\hat{z}\left[-\frac{1}{(s'^2+z^2)^{1/2}}\Bigg|_{s'=0}^{R}\right]=-\frac{\sigma_0 z}{2\epsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}}-\frac{1}{z}\right]\hat{z}$$

$$=\frac{\sigma_0}{2\epsilon_0}\left[1-\frac{1}{\sqrt{1+(R/z)^2}}\right]\hat{z}\quad (\circ)$$

check limiting case:

? What do we expect $\vec{E}$ to be as $z\to\infty$? ($Q=\sigma_0\cdot\pi R^2$)

$$\vec{E}\to\frac{Q}{4\pi\epsilon_0 z^2}\hat{z}=\frac{\sigma_0\pi R^2}{4\pi\epsilon_0 z^2}\hat{z}=\frac{\sigma_0 R^2}{4\epsilon_0 z^2}\hat{z}$$

+ for $R/z \ll 1$, $\frac{1}{\sqrt{1+(R/z)^2}}=(1+(R/z)^2)^{-1/2}\approx 1-\frac{R^2}{2z^2}$

using this relation in $(\circ)$

$$\vec{E}\approx\frac{\sigma_0}{2\epsilon_0}\left[1-\left(1-\frac{R^2}{2z^2}\right)\right]\hat{z}=\frac{\sigma_0 R^2}{4\epsilon_0 z^2}\hat{z}\ ✓$$