Last Time

The divergence thm:

$$\int_\Omega \nabla\cdot\vec{v}\,d\tau = \oint_{\partial\Omega}\vec{v}\cdot d\vec{a}$$

runs into trouble when we evaluate each side w/ $\vec{V} = \frac{1}{r^2}\hat{r}$ for a spherical domain.

$$\oint_{\partial\Omega}\vec{v}\cdot d\vec{a} = 4\pi$$

$\partial\Omega$ ↗ but "$\nabla\cdot\vec{v} = 0$" gives $\int_\Omega \nabla\cdot\vec{v}\,d\tau = 0$

an apparent contradiction

But (v^r)

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(1/r^2)\right) = 0$$

is only true for pts w/ $r \neq 0$.

At $r=0$, $\nabla\cdot\vec{v} = ?$

Dirac Delta Distribution

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{"}\infty\text{"} & x = 0 \end{cases}$$

w/ $\int_{-\infty}^{+\infty}\delta(x)\,dx = 1$, an integrable infinity. What are the units of $\delta(x)$?

$\delta(x)$ can be constructed as the limit of increasing sharply peaked fcns w/ unit area.

properties: 1. $\delta(x) = \delta(-x)$ (even)

2. $\int_{-a}^{b}\delta(x)\,dx = 1$ for $a>0, b>0$

for a function $f(x)$:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{+\infty} f(0)\,\delta(x)\,dx$$

(only val. at 0 will contribute)

$$= f(0)\int_{-\infty}^{+\infty}\delta(x)\,dx$$

(prop. 2 ⟶) $= f(0)$

similarly: (let $y = x-a$)

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = \int_{-\infty}^{+\infty} f(y+a)\,\delta(y)\,dy$$

$$= \int_{-\infty}^{+\infty} f(a)\,\delta(y)\,dy$$

$$= f(a)\int_{-\infty}^{+\infty}\delta(y)\,dy$$

$$= f(a).$$

there is a two-dimensional version:

$\delta^2(x,y) = \delta(x)\delta(y)$ w/ $\int_S \delta^2(x,y)\,dx\,dy = 1$ provided the surface S encloses the origin, else the integral is zero

three dimensional version: $\delta^3(x,y,z) = \delta(x)\,\delta(y)\,\delta(z)$ which we also denote $\delta^3(\vec{r})$, w/

$\int_\Omega \delta^3(\vec{r})\,d\tau = 1$ provided Ω encloses the origin (else zero).

the definition here suggests that $\nabla\cdot\vec{V} \propto \delta^3(\vec{r})$ (for $\vec{v} = \hat{r}/r^2$), & from our calculation of $\oint_{\partial\Omega}\vec{v}\cdot d\vec{a} = 4\pi$,

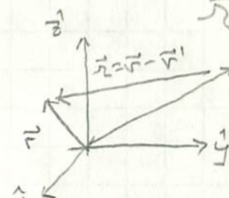
$$\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta^3(\vec{r})$$ saves the divergence thm:

$$\int_\Omega \nabla\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_\Omega 4\pi\delta^3(\vec{r})\,d\tau = 4\pi = \oint_{\partial\Omega}\vec{v}\cdot d\vec{a} \quad ✓$$

"Curly" $\mathscr{r}$

The vector pointing from location $\vec{r}'$ to $\vec{r}$ is defined to be:

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' \quad \text{(check sign by taking } \vec{r}' = 0\ldots)$$



For $\vec{V} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$

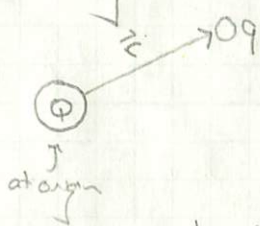
what is $\nabla\cdot\vec{V} = ?$

just a shift of origin here, so

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \delta^3(\vec{\mathscr{r}}) = \delta^3(\vec{r}-\vec{r}').$$

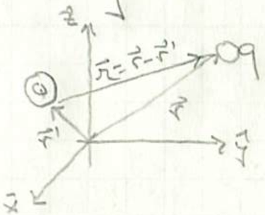
2 Observations

1. For charges Q & q, the electric force on q due to Q is:



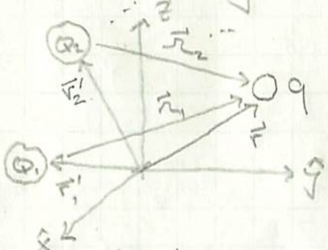
$$\vec{F} = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{r}$$

moving Q to $\vec{r}'$



$$\vec{F} = \frac{Qq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}} = \frac{Qq}{4\pi\epsilon_0 \mathscr{r}^3}\vec{\mathscr{r}}$$

2. For a set of charges $\{Q_j\}_{j=1}^{n}$ at locations $\{\vec{r}_j'\}_{j=1}^{n}$,



$$\vec{F} = q\sum_{j=1}^{n}\frac{Q_j}{4\pi\epsilon_0 \mathscr{r}_j^2}\hat{\mathscr{r}}_j \quad (*)$$

w/ $\vec{\mathscr{r}}_j = \vec{r} - \vec{r}_j'$

the forces add (superposition)

it did not have to be this way

Define $$\vec{E} = \sum_{j=1}^{n}\frac{Q_j}{4\pi\epsilon_0 \mathscr{r}_j^2}\hat{\mathscr{r}}_j \quad (o)$$ the "electric field" - just $(*)$ w/ q chopped off.

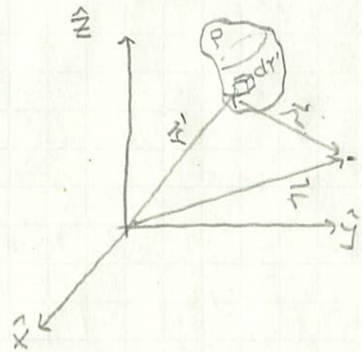
then $\vec{F} = q\vec{E}$.

Continuous Distributions of Charge

$\rho(\vec{r})$ is the charge per unit volume, "charge density"

$\rho = 0$ out here.

we can use observations 1. & 2. to write a continuous version of (o)



In the tiny volume $d\tau'$ at $\vec{r}'$, the total charge is

$$dq = \rho(\vec{r}')\,d\tau'$$

then the contribution to the electric field at $\vec{r}$ is:

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}} \quad \text{from observation 1.}$$

Using observation 2. superposition, to get $\vec{E}$ at $\vec{r}$, we just sum up the $d\vec{E}$

$$\vec{E} = \int d\vec{E} = \int \frac{dq}{4\pi\epsilon_0 \mathscr{r}^2}\hat{\mathscr{r}} = \int_{\text{all space}} \frac{\rho(\vec{r}')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}d\tau'$$

all space ← $\rho = 0$ outside of some region

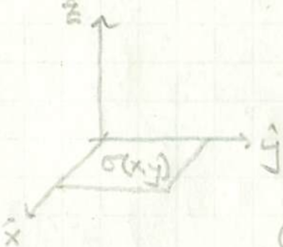
Degenerate Cases

1. surface charge - σ is charge/area

$dq = \sigma(\vec{r}')da'$

$$\vec{E} = \oint_S \frac{\sigma(\vec{r}')\hat{\mathscr{r}}}{4\pi\epsilon_0 \mathscr{r}^2}da'$$

really, surface charge is a volume charge w/ a δ in it – example

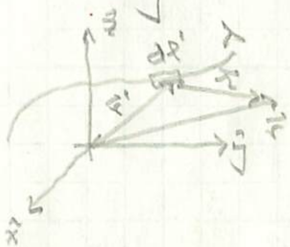


given surface charge $\sigma(x,y)$ the associated volume charge is:

$$\rho(x,y,z) = \sigma(x,y)\,\delta(z)$$

(check units ...)

2. line charge – λ is charge/length



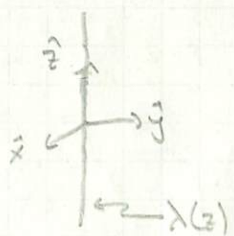
here, $dq = \lambda(\vec{r}')\,d\ell'$

&

$$\vec{E} = \int_{\text{line}} \frac{\lambda(\vec{r}')\,\hat{\mathscr{r}}}{4\pi\epsilon_0\,\mathscr{r}^2}\,d\ell'$$

again, though, there is a ρ w/ 2 δ's that describes the situation!

example: for $\lambda(z)$ lying along the $\hat{z}$ axis, we have



$$\rho(x,y,z) = \lambda(z)\,\delta(x)\,\delta(y)$$

3. The point charge is the last of the reductions from three dimensions, & we know its field from experiment – the volume charge density description is:

$$\rho = Q\,\delta^3(\underbrace{\vec{r}-\vec{r}'}_{=\vec{\mathscr{r}}}) \quad \text{for a pt charge at } \vec{r}'$$