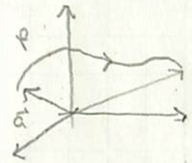
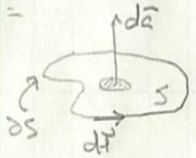
Last Time

Different versions of the fund. thm. of Calculus:



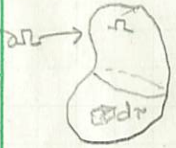
$$\int_a^b \nabla f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$$

=



$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_{\partial S} \vec{v} \cdot d\vec{l}$$

=

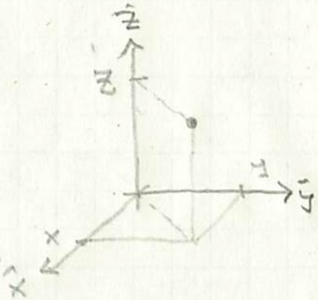


$$\int_\Omega \nabla \cdot \vec{v}\, d\tau = \oint_{\partial\Omega} \vec{v} \cdot d\vec{a}$$

w/ Cartesian coords, we can only integrate over boxey shapes...

Curvilinear Coordinates

Cartesian coords: x, y, z



basis vec.s point in dir. of increasing coord. value

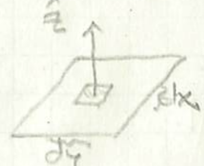
basis vec.s are $\perp$ at every pt.

$\{\hat{x}, \hat{y}, \hat{z}\}$ point in the same direction everywhere.

Infinitesimal vector $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

Volume elt: $d\tau = dx\,dy\,dz$

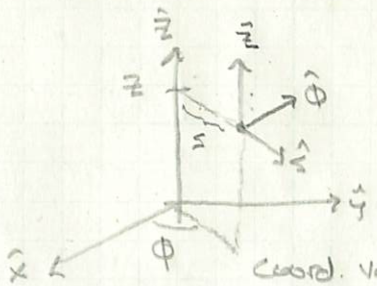
surface elt.s



$d\vec{a} = dx\,dy\,\hat{z}$ etc

=

Cylindrical coord.s: s, ϕ, z

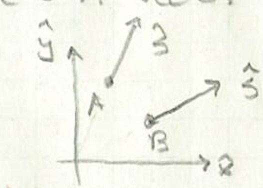


$s = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}(y/x)$

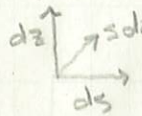
basis vec.s point in dir. of increasing coord. value, & are $\perp$ at every point.

but now, the basis vectors depend on position:



$\hat{s}$ at A not equal to $\hat{s}$ at B.

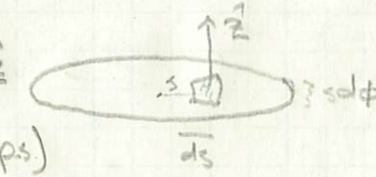
Infinitesimal vector:



$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

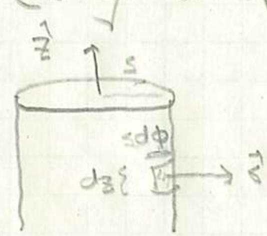
volume elt. $d\tau = (ds)(s\,d\phi)(dz)$

area elt.s:



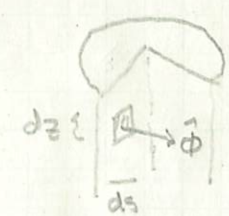
$d\vec{a} = (ds)(s\,d\phi)\hat{z}$

(for cylinder tops)



$d\vec{a} = (s\,d\phi)\,dz\,\hat{s}$

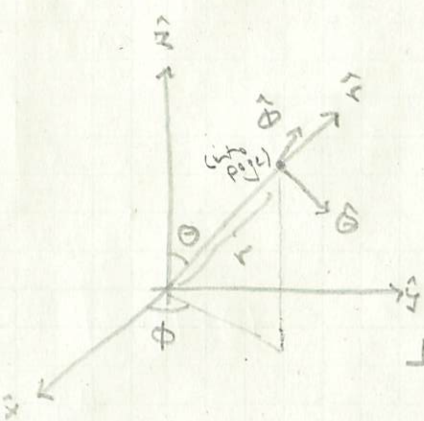
(for cylindrical sides)



$d\vec{a} = (ds)(dz)\hat{\phi}$

(not used much)

Spherical coord.s



$r = \sqrt{x^2+y^2+z^2}$

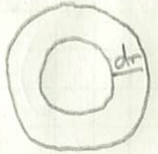
$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\phi = \tan^{-1}(y/x)$

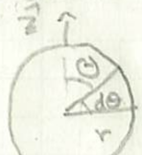
basis vecs point in the direction of increasing coordinate value, & are $\perp$ at every point.

infinitesimal vector: $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

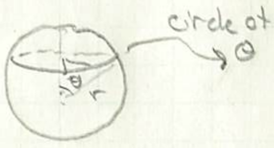
aside: make small moves in each direction independently:



$d\vec{l} = dr\,\hat{r}$,



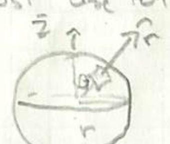
$d\vec{l} = r\,d\theta\,\hat{\theta}$



$d\vec{l} = r\sin\theta\,d\phi\,\hat{\phi}$

$d\tau = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$ is the volume elt

The most useful surface elt. is



$d\vec{a} = (r\,d\theta)(r\sin\theta\,d\phi)\,\hat{r}$.

Divergence of $\hat{r}/r^2$

Each coord. system has its own ∇, $\nabla\cdot$, $\nabla\times$ (see front cover of your book).

Let's work out the divergence of $\vec{v} \equiv \hat{r}/r^2$ in spherical coord.s:

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r}$

For spherical coordinates: $\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v^r)$

v^r: r-comp. of $\vec{v}$, $v^r = r^{-2}$

$= 0$ as we saw before, except possibly at $r \to 0$

Consider the divergence theorem applied to a ball of radius R

$$\int_\Omega \nabla\cdot\vec{v}\,d\tau = \oint_{\partial\Omega} \vec{v}\cdot d\vec{a}$$

w/ $d\vec{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$, &

$\vec{v}\big|_{\partial\Omega} = \hat{r}/R^2$

so that $\oint_{\partial\Omega} \vec{v}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} \sin\theta\,d\theta\,d\phi = 4\pi$

but on the left, w/ $\nabla\cdot\vec{v} = 0$, we appear to get zero, ... ?!

Dirac Delta Distribution

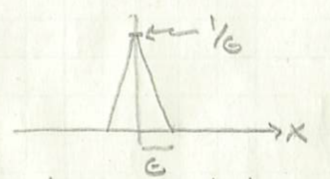
We need a way to describe an object that is zero everywhere except at a point, where it is "infinity" (or "undefined") but that is integrable.

Dirac delta distribution:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{"}\infty\text{"} & x = 0 \end{cases}$$

w/ $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$ (integrable)

"construction"



let $\delta_\epsilon(x)$ be a triangle centered at $x=0$, w/ base 2ϵ, & height $1/\epsilon$, then $\forall \epsilon$, the area under $\delta_\epsilon(x)$ is 1, take

$$\delta(x) \equiv \lim_{\epsilon \to 0} \delta_\epsilon(x)$$

properties: 1. $\delta(-x) = \delta(x)$ (symmetric)

2. $\int_{-\infty}^{+\infty} \delta(x)\,dx = \int_{-a}^{b} \delta(x)\,dx = 1 \quad \forall\, a,b>0$

3. given a function $f(x)$,

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\,dx = \int_{-\infty}^{+\infty} \delta(x) f(0)\,dx = f(0) \overbrace{\int_{-\infty}^{+\infty} \delta(x)\,dx}^{=1 \text{ by def.}} = f(0)$$

$\delta(x) f(x)$ is non-zero only at $x=0$

there is a 2-dimensional version: $\delta^2(x,y) \equiv \delta(x)\delta(y)$

w/

$$\int_S \delta^2(x,y)\,da = 1$$ provided the surface encloses the origin.

& a 3-dimensional version: $\delta^3(x,y,z) = \delta(x)\delta(y)\delta(z)$

w/

$$\int \delta^3(x,y,z)\,d\tau = 1$$ provided the volume encloses the origin.

this last $\int$ suggests that we have

$$\nabla \cdot \frac{\hat{r}}{r^2} \propto \underbrace{\delta^3(x,y,z)}_{\text{sometimes written } \delta^3(\vec{r})}$$

from our div. thm. investigation, we expect

$$\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^3(\vec{r})$$