Presentation Problem 1

J. Franklin

August 28, 2026

Problem (as given in assignment)

Use the gradient operator,

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z},$$

to generate the basis vectors in spherical coordinates.

Presentation

In this problem, we are asked to use the gradient operator, expressed in Cartesian coordinates,

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{1}$$

to develop the unit basis vectors in spherical coordinates, $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}\}$.

The gradient is a useful tool for this problem because when it is applied to a scalar function like $f(x, y, z)$, it returns the direction of greatest increase of f. That characteristic feature of the gradient can be established quickly using Taylor expansion: For a point $\{x, y, z\}$, we know the value of f. At a nearby point, displaced from the original by (small) dx, dy and dz, the value of f is approximately:

$$f(x + dx, y + dy, z + dz) \approx f(x, y, z) + dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y} + dz\frac{\partial f}{\partial z}, \tag{2}$$

or, defining $d\boldsymbol{\ell} \equiv dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$,

$$f(x + dx, y + dy, z + dz) \approx f(x, y, z) + d\boldsymbol{\ell} \cdot \nabla f(x, y, z), \tag{3}$$

and the way to maximize the dot product is to take $d\boldsymbol{\ell} \parallel \nabla f(x, y, z)$.

In any coordinate system, the basis vectors point in the direction of increasing coordinate value. For the function $r(x, y, z) \equiv \sqrt{x^2 + y^2 + z^2}$ that defines the radial coordinate, the direction of greatest increase, at $\{x, y, z\}$, is $\nabla r(x, y, z)$, all we have to do is calculate the derivatives:

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2} 2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \tag{4}$$

with the y and z derivatives obtained by taking $x \to y$ and $x \to z$ respectively. The gradient of r is then

$$\nabla r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \right]. \tag{5}$$

We want $\hat{\mathbf{r}}$, the *unit* vector pointing in the direction of greatest increase for the coordinate r, for the vector $\nabla r(x, y, z)$ the magnitude (squared) is $\nabla r(x, y, z) \cdot \nabla r(x, y, z) = 1$, it is already a unit vector, so

$$\hat{\mathbf{r}} = \nabla r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left[x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \right]. \tag{6}$$

Moving on to the angular pieces, we have $\phi = \tan^{-1}(y/x)$. Our first job is to identify the derivative of the arctangent. One useful trick is to note that

$$\psi = \tan^{-1}(\tan \psi), \tag{7}$$

then let $u \equiv \tan \psi$. Taking the u-derivative of the equation that defines u gives[1]

$$1 = \frac{d \tan \psi}{d\psi} \frac{d\psi}{du} \longrightarrow \frac{d\psi}{du} = \cos^2 \psi = \frac{1}{1 + u^2}. \tag{8}$$

Now taking the u derivative of both sides of (7), we have

$$\frac{d\psi}{du} = \frac{d \tan^{-1}(u)}{du} = \frac{1}{1 + u^2}. \tag{9}$$

Returning to ϕ, with this result, the x and y derivatives are

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2} \qquad \frac{\partial \phi}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}. \tag{10}$$

The gradient of ϕ is, then,

$$\nabla \phi = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{x^2 + y^2}. \tag{11}$$

Normalizing to get $\hat{\boldsymbol{\phi}}$ gives

$$\nabla \phi \parallel \hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}. \tag{12}$$

[1] From $u \equiv \tan \psi$, note that $u^2 = (1 - \cos^2 \psi)/\cos^2 \psi$, then we can isolate: $\cos^2 \psi = 1/(1 + u^2)$.

Finally, for $\theta \equiv \tan^{-1}(\sqrt{x^2+y^2}/z)$, we just have more application of the chain rule. The derivatives are

$$\frac{\partial\theta}{\partial x} = \frac{xz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} \qquad \frac{\partial\theta}{\partial y} = \frac{yz}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)} \qquad \frac{\partial\theta}{\partial z} = -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}, \tag{13}$$

and the gradient is

$$\nabla\theta = \frac{xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - (x^2+y^2)\hat{\mathbf{z}}}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}. \tag{14}$$

Normalizing gives

$$\nabla\theta \parallel \hat{\boldsymbol{\theta}} = \frac{xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - (x^2+y^2)\hat{\mathbf{z}}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}. \tag{15}$$

None of these expressions for the spherical basis vectors look at all familiar, but that is because they are written in Cartesian coordinates. Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, we recover

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}. \end{aligned} \tag{16}$$

It is interesting to note that the gradient's role in determining directions of increase holds even in spaces that are not "flat" (ones that have a different definition of length than the usual Pythagorean one). How might we determine the relevant unit basis vectors in a setting that was not just a coordinate transformation away from Cartesian?