Problem 1

Let $x \equiv \sin\theta$, then taking the θ derivative of $\theta = \sin^{-1}(x(\theta))$ gives:

$$1 = \frac{d\sin^{-1}(x)}{dx} \cdot \frac{dx}{d\theta} \quad \& \quad \frac{dx}{d\theta} = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

$$= \frac{d\sin^{-1}(x)}{dx} \cdot \sqrt{1-x^2} \Rightarrow \frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Problem 2

Using the result from Problem 1: $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{d}{dx}\left(\sin^{-1}(x)\right)dx = \sin^{-1}(x)$

Using substitution: let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$ & $\sqrt{1-x^2} = \cos\theta$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta = \sin^{-1}(x).$$

Problem 3

Take $x(t) = Ae^{ut}$, & use this ansatz in $\ddot{x}(t) = -\omega^2 x(t) - 2b\dot{x}(t)$,

$$u^2 Ae^{ut} = -\omega^2 Ae^{ut} - 2bu Ae^{ut} \Rightarrow u^2 + 2bu + \omega^2 = 0$$

so

$$u = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2-\omega^2}$$

For underdamped motion, $b^2 - \omega^2 < 0$, so $u = -b \pm i\sqrt{\omega^2 - b^2}$, & the general solution is:

$$x(t) = e^{-bt}\left[A\cos\left(\sqrt{\omega^2-b^2}\, t\right) + B\sin\left(\sqrt{\omega^2-b^2}\, t\right)\right]$$

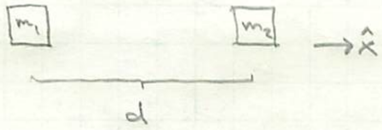
Imposing the initial conditions: $x(0) = A = 0$, & then

$$\dot{x}(t) = -be^{-bt} \cdot B\sin\left(\sqrt{\omega^2-b^2}\, t\right) + e^{-bt} B\sqrt{\omega^2-b^2}\cos\left(\sqrt{\omega^2-b^2}\, t\right)$$

with $\dot{x}(0) = B\sqrt{\omega^2-b^2} = v_0 \Rightarrow B = \frac{v_0}{\sqrt{\omega^2-b^2}}$, so that:

$$x(t) = \frac{v_0 e^{-bt}}{\sqrt{\omega^2-b^2}} \sin\left(\sqrt{\omega^2-b^2}\, t\right)$$

Problem 4



The force on m_1 is: $\vec{F}_1 = \frac{Gm_1m_2}{d^2}\hat{x}$

& from $\vec{F}_1 = m_1\vec{a}_1$, we have: $\vec{a}_1 = \frac{Gm_2}{d^2}\hat{x}$

similarly, the force on m_2 is: $\vec{F}_2 = -\frac{Gm_1m_2}{d^2}\hat{x}$ w/ $\vec{a}_2 = -\frac{Gm_1}{d^2}\hat{x}$

For $m_2 = -m_1$, we have initial accelerations: $\vec{a}_1 = -\frac{Gm_1}{d^2}\hat{x} = \vec{a}_2$.

Both masses move to the left w/ constant acceleration, they remain the same distance apart (b/c of the identical acceleration).

If m_1 is at 0 initially, & m_2 is at d, the positions are

$$\vec{x}_1 = -\tfrac{1}{2}\left[\frac{Gm_1}{d^2}\right]t^2\hat{x} \quad \text{and} \quad \vec{x}_2 = \left[d - \tfrac{1}{2}\left[\frac{Gm_1}{d^2}\right]t^2\right]\hat{x}$$

The total energy of the system is: $E = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - G\frac{m_1m_2}{d}$

w/

$$\dot{x}_1^2 = \left(\frac{Gm_1}{d^2}\right)^2 = \dot{x}_2^2, \text{ so } \quad E = \tfrac{1}{2}(m_1 - m_1)\dot{x}_1^2 - G\frac{m_1m_2}{d}$$

$$= -G\frac{m_1m_2}{d}, \text{ a constant of the motion}$$

Problem 5

a. $I = \int \frac{dx}{1-x^2}$ note that $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$ and $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$

if we take $a = (1-x)$ & $b = (1+x)$, then $\frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{(1-x)(1+x)}$, so

$$I = \int \frac{dx}{(1-x)(1+x)} = \frac{1}{2}\left[\int \frac{dx}{1-x} + \int \frac{dx}{1+x} \right]$$

&

$$\int \frac{dx}{1-x} \underset{u=1-x}{=} -\int \frac{du}{u} = -\log u = -\log(1-x), \text{ sim. } \int \frac{dx}{1+x} \overset{u=1+x}{=} \int \frac{du}{u} = \log(1+x)$$

so

$$I = \frac{1}{2}\left[\log(1+x) - \log(1-x) \right] = \log\left(\sqrt{\frac{1+x}{1-x}} \right)$$

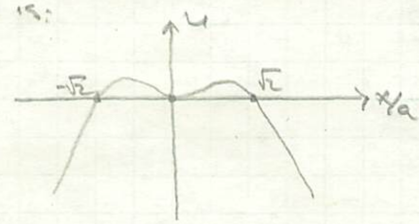
b. $$I = \int_0^x \frac{d\bar{x}}{1-(\bar{x}/a)^2} \underset{z=\bar{x}/a}{=} + \int_0^{x/a} \frac{a\,dz}{1-z^2} = +a\log\left(\sqrt{\frac{1+(x/a)}{1-(x/a)}} \right)$$

Problem 6 (MT 2.52)

For $U(x) = U_0\left[2(x/a)^2 - (x/a)^4\right] = U_0 (x/a)^2\left[2 - (x/a)^2\right]$

a. $F = -\frac{dU}{dx} = -U_0\left[\frac{4x}{a^2} - \frac{4x^3}{a^4}\right] = 4U_0 \cdot \frac{x}{a^2}\left[\left(\frac{x}{a}\right)^2 - 1\right]$

b. $U = U_0 (x/a)^2\left[2-(x/a)^2\right]$ has zeros at $(x/a)=0$, $(x/a) = \pm\sqrt{2}$, as $(x/a) \to \pm\infty$ $U \to -\infty$, so the sketch is:



c. We have $U' = U_0\left[4x/a^2 - 4x^3/a^4\right]$, so $U'' = U_0\left[4 \cdot 1/a^2 - 12x^2/a^4\right]$ and the "effective spring constant" is (at $x=0$)

$$k = U''(0) = 4\frac{U_0}{a^2}$$

the solution to $m\ddot{x} = -kx \Rightarrow x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right)$ w/ $\omega = \sqrt{\frac{k}{m}}$ so

$$\omega = \sqrt{\frac{4U_0}{ma^2}}$$

d. The maxima of the potential occur at $(x/a) = \pm 1$ (where $F=0$) w/ value: $U(x/a=1) = U_0$ and a particle must have at least this energy to escape: $E = \frac{1}{2}mv_0^2 = U_0$, so

$$v = \sqrt{2U_0/m}$$

e. $E = \frac{1}{2}m\dot{x}^2 + U(x) = U_0 + 0$ (initially) then: $\frac{1}{2}m\dot{x}^2 = U_0\left[(x/a)^4 - 2(x/a)^2 + 1\right]$ or

$$\dot{x}^2 = \frac{2}{m}U_0\left((x/a)^2 - 1\right)^2 \Rightarrow \dot{x} = \sqrt{\frac{2U_0}{m}}\left(1 - (x/a)^2\right)$$ (take the root that gives $\dot{x} > 0$)

$$\frac{dx}{1-(x/a)^2} = \sqrt{\frac{2U_0}{m}}\,dt$$ & integrating both sides gives

$$\int_0^x \frac{dx}{1-(x/a)^2} = \int_0^t \sqrt{\frac{2U_0}{m}}\,dt \Rightarrow a\log\left(\frac{\sqrt{1+x/a}}{\sqrt{1-x/a}}\right) = \underbrace{\sqrt{\frac{2U_0}{m}}}_{\equiv K}\,t$$

solving for $(x/a) \equiv y$, we have $\frac{\sqrt{1+y}}{\sqrt{1-y}} = e^{Kt/a} \Rightarrow 1+y = e^{2Kt/a}(1-y)$

or

$$y\left(1+e^{2Kt/a}\right) = e^{2Kt/a} - 1 \quad \text{or} \quad \frac{x}{a} = -\frac{1-e^{2Kt/a}}{1+e^{2Kt/a}}$$

so $x = -a\dfrac{\left(1-e^{2\sqrt{\frac{2U_0}{m}}\,t/a}\right)}{\left(1+e^{2\sqrt{\frac{2U_0}{m}}\,t/a}\right)}$