A First Course in Modular Forms: Corrections to the Fourth Printing

March 1, 2025

Chapter 1

- Page 7, line 8: Change "*the modular invariant*" to "the *modular invariant*".
- Page 10: Part (e) can be added to Exercise 1.1.7 as follows. "The next section will show that $(2\pi)^{-12}\varDelta$ has integral Fourier coefficients, so for the duration of this exercise, renormalize the discriminant function $\varDelta$ so that its first Fourier coefficient is 1. The integer-valued Fourier coefficient function of $\varDelta$ is called Ramanujan's τ-function. Because the variable of $\varDelta$ is also denoted τ and both usages of τ are time-honored, we accept the notation-collision

  $$\varDelta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n.$$

  Later in this book we will show that the cusp form space $\mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is one-dimensional. Thus $E_{12} - E_6^2 = c\varDelta$ for some $c \in \mathbb{C}$. Show that $c = 24(1/B_6 - 1/B_{12})$. In light of the known value $B_{12} = -691/2730$ and the value $B_6 = 1/42$ from above, we thus have $-24 \cdot 691/B_{12} \in \mathbb{Z}$ and $691c \in -24 \cdot 691/B_{12} + 691\mathbb{Z}$. Explain why the Fourier coefficients of $691(E_{12} - 1)$ and $(-24 \cdot 691/B_{12})\varDelta$ are integers that are congruent modulo 691. Deduce Ramanujan's congruence,

  $$\tau(n) \equiv \sigma_{11}(n) \pmod{691}, \quad n \geq 1."$$

- Page 42, last line of five-line display: change "dQ" to "Q".

Chapter 2

- Page 56, line 4: Change "are a subset" to "form a subset".

Chapter 3

- Page 65: Add a sentence at the end of the first paragraph of Section 3.1, "In particular, $X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^*$ has genus 0 by Lemmas 2.3.1, 2.3.2, and 2.4.1."

- Page 70: Toward the end of exercise 3.1.4(f), the "Show that the 14 points..." sentence should have $j = -6, \dots, 6, 7$ rather than $-6, \dots, 6, \infty$.
- Page 71, line 3: Before "Determine", add a sentence: "The elements of the two-point class are equivalent modulo $\mathrm{SL}_2(\mathbb{Z})$, as are two elements of the four-point class, so the class sizes are consonant with our formulas for order-3 elliptic points."
- Page 80: In the "Conversely…" sentence immediately after the white-space, change "$\omega(f)$" to "$\omega = \omega(f)$".
- Page 87: Change " for $k \geq 4$" to ", even $k \geq 4$" in (3.12).
- Page 87, line (-14): Change "Exercise 3.4.2" to "Exercises 3.3.6 and 3.4.3".
- Page 88: Exercise 3.5.1 repeats Exercise 3.2.4.
- Page 89, line (-7): "Exercise 3.2.1" should be "Exercise 3.2.5".
- Page 91: Incidentally Theorem 3.6.1 shows that $\varepsilon_\infty^{\mathrm{reg}}$ is even.
- Page 92: At the end of the section, add a paragraph as follows: "We sketch why f_1 exists as just described. About any point $p \in \mathcal{H}$ we have $\tilde{f}(\tau) = (\tau - p)^{2n} \tilde{f}_o(\tau)$ where $n \in \mathbb{Z}$ and $\tilde{f}_o$ is analytic and nonzero on an open disk about p. So $\tilde{f}$ has two local square roots on this disk by complex analysis, e.g., $\tilde{f}_o(z) = e^{g_o(z)}$ with g_o determined up to $2\pi i\mathbb{Z}$, and the square roots are $\pm(\tau - p)^n e^{g_o(\tau)/2}$. Now we have two local square roots about each $p \in \mathcal{H}$. Choose one of them about i and denote it f_i. Given any point $\tau \in \mathcal{H}$, we can continue f_i along any path from i to τ by chaining together local square roots on a finite succession of disks. Because the upper half plane is simply connected, a homotopy argument shows that this continuation is path-independent."
- Page 92: The formula in Exercise 3.6.1 should be $\mathrm{div}(\omega) = 2\,\mathrm{div}(f) + k\,\mathrm{div}(d\tau)$.
- Page 92: Add a hint prompt to Exercise 3.6.4.
- Page 94, line 15: Change aI to a in the second line of the display.

Chapter 4

- Page 110, line (-11): Change "Exercise 1.1.2" to "Exercise 1.1.4".
- Page 114: The second line after (4.9) should begin "$n \equiv 0\,(c_v)$".
- Page 129: Add "We will prove the following result in Section 5.2." before Theorem 4.5.2.
- Page 133: Add "We will prove the following result in Section 5.2." before Theorem 4.6.2.
- Page 141: Add "We will prove the following result in Section 5.2." before Theorem 4.8.1.
- In Section 4.8 and Exercise 4.8.6, the claim $G_1^{\overline{(-c_v,d_v)}}(\tau) = -G_1^{\overline{(c_v,d_v)}}(\tau)$ when $\overline{c}_v \neq 0$ is wrong; rather, $G_1^{\overline{(-c_v,-d_v)}}(\tau) = -G_1^{\overline{(c_v,d_v)}}(\tau)$ regardless of whether $\overline{c}_v = 0$; because the two vectors $\overline{\pm(c_v,d_v)}$ describe the same cusp of $\Gamma(N)$, these relations among the Eisenstein series do not support the claimed dimension $\varepsilon_\infty/2$ of the Eisenstein space $\mathcal{E}_1(\Gamma(N))$. See the paper

Moduli interpretation of Eisenstein series by Kamal Khuri-Makdisi (International Journal of Number Theory, vol. 8, no. 3, 2012, pages 715–748), especially Proposition 4.3, for the subtle symmetry of weight 1 Eisenstein series.

- Page 150, before the four-line display: Change "shows that part of the integral is" to "shows that for $\mathrm{Re}(s) > 1$, part of the integral is".
- Page 151, line 1: The exponent of μ_N is also $\langle v, wS \rangle$. This fact combines with line 3 to show that the Fourier transform being used here has order 2 because of how it incorporates S, rather than order 4 as the usual Fourier transform does.
- Pages 152–153: Unlike similar integrals involving $\vartheta - 1$ in the previous section and earlier in this section, the integral in (4.44) converges robustly at both its endpoints, making it entire in s. The manipulations leading from (4.44) to the next display (unlabeled) and (4.45) require $\mathrm{Re}(s) > 1$, but then (4.45) already gives the analytic continuation of the completed Eisenstein series before using the theta transformation law to move the left endpoint to 1. Moving the left endpoint is still necessary to obtain the functional equation, and now it does so without introducing meromorphic terms.

  The assertion immediately before Theorem 4.10.2 that the theorem holds for nonpositive integers k as well is not correctly supported, because of an error in Exercise 4.10.6.
- Page 153, line 2: Change "$g_k^{\overline{v}}(s,\gamma)$" to "$g_k^{a}(s,\gamma)$".
- Page 162, first line of Exercise 4.11.5: "$\dim(\mathcal{S}_1(\Gamma_0(3))$" is missing a right parenthesis.

Chapter 5

- Page 170, line 1: The first complete sentence should start, "Thus for any $\gamma \in \Gamma_0(N) \ldots$".
- Page 173: Immediately before Proposition 5.2.3, change "but also" to "but further, $E_k^{\psi,\varphi,t}$ is a T_p-eigenform for all primes $p \nmid N$, and sometimes for all primes p, as follows."

  Immediately after Proposition 5.2.3, change "(Exercise 5.2.5)" to "(Exercise 5.2.5(a–d))".

  After that paragraph, but before the one-sentence paragraph, "The Hecke operators commute," add the following two paragraphs:

  "Let N and k be positive integers, and let χ be a character modulo N. If $k \geq 3$, let $A_{N,k}(\chi)$ be the set of triples (ψ, φ, t) such that ψ and φ are primitive Dirichlet characters modulo u and v with $(\psi\varphi)(-1) = (-1)^k$ and $\psi\varphi = \chi$ at level N, and t is a positive integer such that $tuv \mid N$. If $k = 2$, stipulate the additional condition $1 < tuv$, ruling out the triple $(\mathbf{1}_1, \mathbf{1}_1, 1)$. If $k = 1$, view the characters ψ, φ of any $A_{N,k}(\chi)$-element as an unordered pair despite the element being written as an ordered triple. With Proposition 5.2.3 in hand, we show that the set

$$\{E_k^{\psi,\varphi,t} : (\psi,\varphi,t) \in A_{N,k}(\chi)\}$$

is linearly independent. To do so, begin by noting the small fact that for $a, b, \tilde{a}, \tilde{b} \in \mathbb{C}^*$, if $ab = \tilde{a}\tilde{b}$ and $a+b = \tilde{a}+\tilde{b}$ then $\{\tilde{a}, \tilde{b}\} = \{a, b\}$; and further, if $|\tilde{a}| = |a|$ and $|\tilde{b}| = |b|$ and these values are distinct then $(\tilde{a}, \tilde{b}) = (a, b)$ (Exercise 5.2.5(e)). Let ψ, φ be the first two entries of an element of $A_{N,k}(\chi)$, and similarly for $\tilde{\psi}$ and $\tilde{\varphi}$. Suppose that $\psi(p)+\varphi(p)p^{k-1} = \tilde{\psi}(p)+\tilde{\varphi}(p)p^{k-1}$ for all primes $p \nmid N$. Bring the fact to bear on Proposition 5.2.3, with $a = \psi(p)$, $b = \varphi(p)p^{k-1}$, $\tilde{a} = \tilde{\psi}(p)$, and $\tilde{b} = \tilde{\varphi}(p)p^{k-1}$ for any such p. For $k > 1$, the proposition and the fact show that $\tilde{\psi}(p) = \psi(p)$ and $\tilde{\varphi}(p) = \varphi(p)$, and so $(\tilde{\psi}, \tilde{\varphi}) = (\psi, \varphi)$ by Dirichlet's theorem on primes in an arithmetic progression. For $k = 1$, the proposition and the fact show that $\{\tilde{\psi}(p), \tilde{\varphi}(p)\} = \{\psi(p), \varphi(p)\}$ for each $p \nmid N$; this doesn't immediately say that one of $\tilde{\psi}, \tilde{\varphi}$ matches ψ at all such p, but a match does hold—if $(\psi/\tilde{\psi})(p) \neq 1$ for some $p \nmid N$, then $(\psi/\tilde{\varphi})(p) = 1$, so $(\mathbb{Z}/N\mathbb{Z})^*$ is the union of $\ker(\psi/\tilde{\psi})$ and $\ker(\psi/\tilde{\varphi})$, making the second kernel all of $(\mathbb{Z}/N\mathbb{Z})^*$ (Exercise 5.2.5(f)). Summarizing so far, the eigenvalues of an Eisenstein series $E_k^{\psi,\varphi,t}$ determine the characters ψ and φ, remembering that the pair is unordered for $k = 1$.

Now consider a collection $\{E_k^{\psi_i,\varphi_i,t_{i,j}}\}$ of Eisenstein series, with each triple $(\psi_i, \varphi_i, t_{i,j})$ in $A_{N,k}(\chi)$ and the pairs (ψ_i, φ_i) distinct. Fix a prime $p \nmid N$. For each i there exists a λ_i such that $T_p E_k^{\psi_i,\varphi_i,t_{i,j}} = \lambda_i E_k^{\psi_i,\varphi_i,t_{i,j}}$ for all j. The previous paragraph has shown that the λ_i are distinct because the character-pairs are distinct. Fix one i, and let $T_{(i)} = \prod_{i' \neq i}(T_p - \lambda_{i'})$. Thus $T_{(i)}$ dilates each $E_k^{\psi_i,\varphi_i,t_{i,j}}$ by the nonzero factor $\prod_{i' \neq i}(\lambda_i - \lambda_{i'})$, which depends on i but not on j, and $T_{(i)}$ annihilates all $E_k^{\psi_{i'},\varphi_{i'},t_{i',j}}$ with $i' \neq i$. Consider a linear relation $\sum_i \sum_j c_{i,j} E_k^{\psi_i,\varphi_i,t_{i,j}} = 0$. For each i, applying $T_{(i)}$ to this relation shows that $\sum_j c_{i,j} E_k^{\psi_i,\varphi_i,t_{i,j}} = 0$. Also, each $E_k^{\psi_i,\varphi_i,t_{i,j}}$ has lowest-order nonconstant term $a_1(E_k^{\psi_i,\varphi_i,1})q^{t_{i,j}}$, making the $E_k^{\psi_i,\varphi_i,t_{i,j}}$ linearly independent, and so $c_{i,j} = 0$ for all j. Thus the linear relation under consideration is trivial. That is, the set of Eisenstein series $E_k^{\psi,\varphi,t}$ with $(\psi, \varphi, t) \in A_{N,k}(\chi)$ is linearly independent, making it a basis of $\mathcal{E}_k(N, \chi)$. This proves Theorems 4.5.2, 4.6.2, and 4.8.1. The decomposition $\mathcal{E}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{E}_k(N, \chi)$ from the end of Section 4.3 shows that the index set $A_{N,k} = \bigcup_\chi A_{N,k}(\chi)$ gives a basis of $\mathcal{E}_k(\Gamma_1(N))$."

- Page 177: Change the beginning of Exercise 5.2.5 to, "Parts (a) through (d) of this exercise prove Proposition 5.2.3. Parts (e) and (f) are used in the discussion of linear independence in the text after the proposition." Add to the exercise,

  "(e) Show that if $a, b, \tilde{a}, \tilde{b} \in \mathbb{C}^*$, with $ab = \tilde{a}\tilde{b}$ and $a + b = \tilde{a} + \tilde{b}$ then $\{\tilde{a}, \tilde{b}\} = \{a, b\}$; further, if $|\tilde{a}| = |a|$ and $|\tilde{b}| = |b|$ and these values are distinct then $(\tilde{a}, \tilde{b}) = (a, b)$.

  (f) Show that no group is the union of two proper subgroups."

- Page 194: To establish Proposition 5.7.7, consider a commutative diagram, in which ι is an injection,

$$\begin{tikzcd} & \bigotimes_i V_i/V_i^{K_i} \\ V^H = \bigotimes_i V_i^{H_i} \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \bigotimes_i V_i^{H_i}/V_i^{\langle H_i,K_i\rangle}. \arrow[uu, "\iota"'] \end{tikzcd}$$

  This gives the result,

$$V^H \cap \sum_i V^{K_i} = \ker(f) = \ker(g) = \sum_i V^{\langle H,K_i\rangle}.$$

  Exercises 5.7.7 and 5.7.8 are now irrelevant and should be canceled.
- Page 200: Add a sentence at the end of Exercise 5.8.6(b): "Note further that the set $\{f(n\tau) : nM \mid N\}$ is linearly independent, because the lowest-order term of each $f(n\tau)$ is q^n."
- Page 201: Although Proposition 5.9.1 is correct, the claim in its proof that $|a_n| \leq Cn^{k-1}$ for Eisenstein series is true only for $k \geq 3$, and the same error repeats three lines after the end of the proof. For $k = 1, 2$, the estimate is $|a_n| \leq C_\epsilon n^{k-1+\epsilon}$ for any $\epsilon > 0$
- Page 207: Exercise 5.9.1(b) works only for $k \geq 3$. For $k = 1$ and $k = 2$, what can be shown is $|a_n| \leq Cn^{k-1+\varepsilon}$ for any $\varepsilon > 0$. This suffices for Proposition 5.9.1. For $k = 2$, note that $\sigma_1(n) = \sum_{d|n}(n/d) = n\sum_{d|n} 1/d$ is at most $n(1+\ln n)$, and this is at most $Cn^{1+\varepsilon}$. For $k = 1$, let $\varepsilon > 0$ be given; for each prime p there exists some $C_{\varepsilon,p}$ such that $\sigma_0(p^e) = e+1 \leq C_{\varepsilon,p}p^{e\varepsilon}$ for $e = 1, 2, 3, \ldots$, and further we may take $C_{\varepsilon,p} = 1$ for all large enough p, so altogether $\sigma_0(n) \leq C_\varepsilon n^\varepsilon$ where $C_\varepsilon = \prod_p C_{\varepsilon,p}$. In fact, once this result for $k = 1$ is established, for any $k \geq 2$ we have $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \leq \sum_{d|n} n^{k-1} = n^{k-1}\sigma_0(n) \leq C_\varepsilon n^{k-1+\varepsilon}$. This suffices for Proposition 5.9.1 but is weaker than necessary for $k \geq 3$.

Chapter 6

- The Riemann surface trace operator $\varGamma_1\backslash\mathcal{H}^* \longrightarrow \varGamma_2\backslash\mathcal{H}^*$ of section 6.2 differs by a factor of 2 from the related group double coset trace operator elsewhere in the book if $-I$ lies in $\varGamma_2$ but not in $\varGamma_1$. This condition does not arise in any situation where we conflate the two.

Chapter 7

- Page 299, line (-2): Change $\tau_{P+P'}$ to $\tau^*_{P+P'}$.

- Page 310, line (-2): Change "Theorem 6.3.2" to "Proposition 6.3.2".

Chapter 8

- Page 347: Change $h_{(0)}$ to h in the last display.

Chapter 9

- Page 371: Near the bottom of the page, change "an elliptic curve" to "an elliptic curve E", and change "a modular curve" to "a modular curve X".
- Page 379: The fifth line of Section 9.2 should say "the points of order dividing ℓ^n".
- Page 392, line (-5): Change "Let let" to "Let".
- Page 403, line (-7): Change X to x.

Hints and Answers to the Exercises

- Page 417, line 2: Change E_4 to G_4.
- Page 417: Add a hint to Exercise 3.6.4, as follows. "For $k = 1$, the divisor $\mathrm{div}(\omega)$ is canonical and $\lfloor \mathrm{div}(f) \rfloor = (1/2)\mathrm{div}(\omega) + \sum_i (1/2)x_i$ and $\lfloor \mathrm{div}(f) - \sum_i x_i - \sum_i (1/2)x_i' \rfloor = (1/2)\mathrm{div}(\omega) - \sum_i (1/2)x_i$. Use the Riemann–Roch Theorem and its corollary."
- Page 419, line (-7): Change "$\frac{1}{N}\cot(\frac{\pi n}{N})$" to "$\frac{\pi}{N}\cot(\frac{\pi n}{N})$".
- Page 421: The Ex. 5.2.5(d) hint should give $a_n(E) = 2(\sigma_1(n) - t\sigma_1(n/t))$, $a_n(T_pE) = 2(\sigma_1(np) + \mathbf{1}_N(p)p\sigma_1(n/p) - t(\sigma_1(np/t) + \mathbf{1}_N(p)p\sigma_1(n/(tp))))$.
- Page 428, line (-2): Change 0_E to 0.
- Page 431, solution to 9.3.2: Change "for some maximal ideal of $\mathcal{O}_{\mathbb{F}}$" to "for some maximal ideal $\mathfrak{p}_{\mathbb{F}}$ of $\mathcal{O}_{\mathbb{F}}$".

Index

- Page 444: Add "reduction of an elliptic curve over $\mathbb{Q}$, bad, semistable, 327" to the index.
- Page 445: Add "semistable reduction of an elliptic curve over $\mathbb{Q}$, 327" to the index.