A First Course in Modular Forms: 
Corrections to the Third Printing

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(The corrections here are also corrections to the earlier printings, but the third printing’s pagination has changed a bit. In case of problems locating a correction here in an earlier printing, please email jerry@reed.edu.)

Chapter 2

• Page 47, lines (−2)–(−1): Change “group” to “subset of SL₂(𝐑)” on line (−2), and change “group” to “subgroup” on line (−1).
• Page 55, line 2: Change “[(a b)]” to “γ = [a b].”
• Page 55, line (−6): Change “proving (1)” to “proving (c)”.
• Page 56, exercise 2.3.2: Change “If” to “If the nontrivial transformation”.
• Page 56, exercise 2.3.5(b): Change “third” to “fourth”.
• Page 61, line 14: Change “width” to “period”.

Chapter 3

• Page 65, line (−2): Change “Y\h(E)” to “Y\f(E)”.
• Page 66, line 20: Change “equal genus” to “equal genus g ≥ 1”.
• Page 69, lines 4–5 (and the relevant bibliography item): Helena Verrill’s fundamental domain drawer is at
  http://www.math.lsu.edu/~verrill/fundomain/
on April 8, 2008.
• Page 70, line (−14): Change “of order 4” to “with j′ ≠ j”.
• Page 70, line (−6): Change “−6, . . . , 7” to “−6, . . . , 6, ∞”.
• Page 70, line (−5): Change “of order 3 or 6” to “with j′ ≠ j”.
• Page 70, line (−1): Change “with with” to “with”.
• Page 72: The quantity denoted h in lines 4–7 should be given a different name such as ĥ, as it is not necessarily the h or the h′ in the discussion on page 74. The sentence “Thus f has period ĥ.” on lines 4–5 is correct, but ĥ need not be the smallest period of f.
• Page 74, line (−9): Change “qₜ = e^{2πi/ĥ′}” to “qₜ = e^{2πi/ḥ′}.”
• Page 74, line (−8): Delete “$q_{h'} = e^{2\pi i r/h'}$ and”.
• Pages 74–75: The discussion in the “Defining...” paragraph on page 74 has an error: the period is $2h$ in the third case independently of $k$, even though $f(\tau + h) = f(\tau)$ for $k$ even. That is, in the first two cases we have $h' = 2h = 2h$ but in the third case we have $h' = h = 2h$. On page 75, remove “and $k$ is odd” from (3.3), and change the text immediately following, from “This can be half-integral in the exceptional case, when $\pi(s)$ or $s$ itself is called an irregular cusp of $\Gamma$.” For example, when $k$ is odd $1/2$ is an irregular cusp...” to “This is half-integral if $(\alpha^{-1} \Gamma \alpha)_\infty = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ (when $\pi(s)$ or $s$ itself is called an irregular cusp of $\Gamma$) and $k$ is odd. For example, $1/2$ is an irregular cusp...

• Page 81, paragraph beginning “On the other hand...”: Replace the discussion leading up to (3.7) with “On the other hand, if $U_j$ contains a cusp $s_j$ then $\delta_j$ takes $s_j$ to $\infty$ and the function $(f[\alpha]_{2n})(z)$ takes the form $g_j(q_h)$ where $h$ is the width of $s$ and $q_h = e^{2\pi iz/h}$; here $g_j$ is meromorphic in $q_h$ at $0$ if the cusp is regular and $g_j$ is meromorphic in $q_h^{1/2} = e^{\pi iz/h}$ at $0$ if the cusp is irregular, but we think of $g_j$ as a series in powers of $q_h$ (half-integral powers in the irregular case) so that the order is the index of the leading coefficient. The relevant local differential is now”.
• Page 81, line (−7): Change “$\mathbb{Q}^{k/2}(\mathcal{H})$” to “$\mathbb{Q}^{\delta k/2}(\mathcal{H})$”.
• Page 90: Change “$\varepsilon_{3,1}$” to “$\varepsilon_3$” on the first line of the three-line display.
• Page 95, line 4: Change “$\gamma^\delta$” to “$\gamma_j$”.
• Page 95, line 15: Change “$\gamma = \gamma = \det m^3$”.
• Section 3.7: A more transparent approach comes from the moduli space point of view, identifying $Y_0(N)$ and $S_0(N)$ as in Theorem 1.5.1. For $N = 1$, elliptic points of $Y_0(N)$ correspond to elliptic curves $\mathbb{C}/\Lambda$, with automorphisms other than multiplication by $\pm 1$. Since only two imaginary quadratic orders have more than two units, and they are both PID’s, there are two elliptic points: one of order 2 corresponding to $\Lambda_\varepsilon = (\mathbb{Z} \oplus \mathbb{Z})$, and one of order 3 corresponding to $\Lambda_\varepsilon = \mu_3 \mathbb{Z} \oplus \mathbb{Z}$. For $I_0(N)$, reason likewise. To find elliptic points of order 3, for example, look at the order $N$ cyclic subgroups of the lattice $\mu_3 \mathbb{Z} \oplus N \mathbb{Z}$, and count how many of them are invariant under multiplication by $\mu_3$. These are precisely the subgroups generated by $m \mu_3 + 1$ where $m^2 - m + 1 \equiv 0 \pmod{N}$. Thus the number of elliptic points of order 3 is the number of solutions of the congruence $m^2 = m + 1 \equiv 0 \pmod{N}$.
• Page 103, line 9: Change “$y_0 \equiv c' c^{-1} \pmod{N}$” to “$y_0 \equiv c' c^{-1} \pmod{N/d}$”. A procedure to list the cusps of $\Gamma_0(N)$ is as follows: For each positive divisor of $N$ choose some nonnegative integer $c$ such that $\gcd(c, N)$ is the given divisor (e.g., take $c = 0$ if the divisor is $N$ and otherwise take $c$ to be the divisor), then for each class in $\mathbb{Z}/\gcd(c, N, \gcd(c, N)) \mathbb{Z}$ choose a representative $a$ coprime to $c$ and take a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Note that $\mathbb{Z}/\mathbb{Z}$ is not empty but rather consists of one class, all of $\mathbb{Z}$. Especially, if $\gcd(c, N, \gcd(c, N)) = 1$ then the only corresponding cusp is $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.
(though for $c = 1$ the representative $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is preferable to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$), and thus a squarefree level $N = p_1 \cdots p_k$ has $2^k$ cusps. Similarly, for $N = 4$ there are three cusps, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and for $N = 9$ there are four cusps, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

- Page 106, fifth line of section 3.9: Change "$D$" to "Div" four times.
- Page 123, line 4: Change $\psi(-1) g(\hat{\varphi}) / v$ to $\psi(-1) g(\hat{\varphi}) / v$.
- Page 123, line 13: Change "$e' \equiv (e + c' b_\gamma) d_\gamma (\text{mod } u)$" to "$e' \equiv (e + c' b_\gamma) d_\gamma - q (\text{mod } u)$", where $q = (d' - dd_\gamma) / v$.
- Page 123, line 12: Change "$t > 0$" to "$t < 0$".
- Page 123, exercise 4.4.1(c): Change "$\Re(s) > 1'$ to "$\Re(s) > 0$".
- Page 127, line 13: Change "$\int_{t=-\infty}^{0}$" to "$\int_{t=0}^{-\infty}$".
- Page 127, line 12: Change "$e' \equiv (e + c' b_\gamma) d_\gamma - q (\text{mod } u)$", where $q = (d' - dd_\gamma) / v$.
- Page 130, line 8: Change "$a_n(k)$" to "$a_n-1(k)$".
- Page 131: On the second line of the first two-line display the summand should begin "$m \mu_N^{m,n}$" rather than "$\mu_N^{m,n}$". On the third line of the three-line display a right parenthesis is missing from "$(1 - \delta(\tau_n))$" and the summand has the same error.

- Page 136, line 11: Change "negated" to "preserved".
- Page 136, line 12: Change "$t > 0$" to "$t < 0$".
- Page 136, line 9: Change "$(n)$" to "$a(k)$".
- Page 140, line 8: Change "$N - c_v$" in the first superscript. Make the same change on page 142 in exercise 4.8.6.
- Page 155, line 5: Change "$\varphi(-1) g(\hat{\varphi}) / v$" to "$\varphi(-1) g(\hat{\varphi}) / v$".
- Page 155, line 11: Change "$(-1)^k$" to "$\psi(-1)$".
- Page 155, line 19: Remove "$\varphi(-1)$".

Chapter 5

- Page 174, diagram (5.8): Change "$D$" to "Div" four times.
- Page 186, line 4: Change "$\beta_j' = \det(\beta) \beta^{-1} v$" to "$\beta_j' = \det(\beta_j) \beta_j^{-1} v$".
- Page 192, line 8: Delete "$\pi_{d_1 d_2} = v$".
- Page 202, third line of the three-line display in the middle of the page: Change "$p_1^{-k-2s}$" to "$p_1^{k-1-2s}$".
- Page 204, second line of section 5.10: Change "$n^s$" to "$n^{-s}$".
- Page 204, line (−3): Change "idempotent" to "an involution".
- Page 205, line 4: Delete "under the Hecke algebra".
- Page 206, line 3: Change "idempotent" to "an involution".
- Section 5.11: The calculation of orthogonality is formally correct, but the absolute convergence of the double integral is not supported correctly by the text.
- Page 209, line 1: Change $f(\alpha(\tau'))$ to $\langle f(\alpha_k) \rangle(\tau')$.
- Page 209, lines 2–3: Delete "if $\Re(k + 2s) > 0$".
Chapter 6

- Page 212, lines (−4) and (−5): Change “$V_{1,2}$” to “$V_1$” and change “$V_{2,1}$” to “$V_2$”.
- Page 215, line (−3): Change “homomorphic” to “homomorphism”.
- Page 220, line (−5): Change “$\gamma$” to “$\delta$”.
- Page 228, diagram (6.11): Change the last “$X$” to “$Y$”.
- Page 233, line (−3): Change “characteristic” to “minimal”.
- Page 235, paragraph starting “Again suppose”: Also $A$ and $k$ are assumed to be structurally compatible as needed.
- Page 239, second display: Change “$g$” to “$g_i$” on the right side of the equality.

Chapter 7

- Page 283, lines 8–10: Replace the sentence beginning, “A complementary argument…” with “For each $\tau \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^2$, the function $f_0^{3\tau}$ determines two $N$-torsion points of $E_j$ unless $2v = 0$, in which case it determines one (Exercise 7.5.5(a)), and so we have found all $N^2$ points of $E_j[N]$ (Exercise 7.5.5(b)).”
- Page 286: Replace Exercise 7.5.5. The new exercise is, “(a) Show that for each $\tau \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^2$, the function $f_0^{3\tau}$ determines two $N$-torsion points of $E_j$ unless $2v = 0$, in which case it determines one. (b) Show that consequently, regardless of whether $N$ is odd or even, we have found all $N^2$ points of $E_j[N]$.”
- Page 300: In the second paragraph of Section 7.7, change “three” to “two” and remove the references to $K_0'$, $C(j,jN)$, and $K_1'$. (It takes some work to show that $K_0'$ is an intermediate field as claimed, and we do not need this result.)
- Page 301, line 14: Change “indeterminants” to “indeterminates”.
- Page 301, line (−6): Change “either $f_0$ or $j_N$” to “$f_0$”.
- Page 304, line (−9): Change “$K_0$, $K_1'$, and $K_1$” to “$K_0$ and $K_1$”.

Chapter 8

- Page 316, line (−15): Change “lie in $k$. For char($k$) = 2, assume that every element of $k$ is a square.”.
- Page 326, line 1: Change the initial value “$a_1(E) = 1$” to “$a_1(E) = 2$”. Furthermore, the normalized solution-counts that are denoted $a_{\rho'}(E)$ on page 325 should be given a different name, as the true $a_{\rho'}(E)$ are indeed defined as on page 361 by the same initial value and recurrence as the Fourier coefficients $a_{\rho'}(f)$ of a newform. For now the normalized solution-counts are renamed $t_{\rho'}(E)$. Note that $t_{\rho}(E) = a_{\rho}(E)$.
- Page 334, line 3: Change “kernel” to “kernel zero and”.
- Page 335, line 12: Change “[N]” to “[p]” twice.
- Page 336, line 3: Change “$\mu_0^N$” to “$\mu_0^N$”. 


• Page 346, line 5: Change “$I \mapsto IM$” to “$I \mapsto I[k|_\rho]$”.

• Page 351, lines 4–6 (clarification, not correction): The argument given is necessary. The fact that the bottom arrow of the diagram is the zero map does not immediately show that $\ker(\tilde{\psi}) = \tilde{E}'[p]$, because the domain of $\psi$ is all of $\tilde{E}$.

• Page 353, line (−8): Change “(7.18)” to “(7.18) (page 304)”.

• Page 361: The right idea is to define for any prime $p$ the local counting zeta-function of $E$, encoding the normalized solution-counts $t_{p^e}(E) = p^e + 1 - |\tilde{E}(\mathbb{F}_{p^e})|$, as

$$Z_p(X, E) = \exp \left( \sum_{e=1}^{\infty} \frac{t_{p^e}(E)}{e} X^e \right).$$

Taking logarithmic derivatives shows that in fact for $X = p^{-s}$ the local zeta-function takes the form of an Euler factor,

$$Z_p(p^{-s}, E) = (1 - a_p(E)p^{-s} + 1_E(p)p^{1-2s})^{-1}.$$

(The relation $a_p(E) = t_p(E)$ is explained in the correction to page 326.) The Hasse-Weil $L$-function of $E$ is the product of these Euler factors,

$$L(s, E) = \prod_p (1 - a_p(E)p^{-s} + 1_E(p)p^{1-2s})^{-1}.$$ 

By the methods of the proof of Theorem 5.9.2, the Dirichlet series form of the $L$-function is

$$L(s, E) = \sum_{n=1}^{\infty} a_n(E)n^{-s}$$

where similarly to the Fourier coefficients of a newform, the $a_n(E)$ satisfy

$$a_1(E) = 1,$$

$$a_p(E) = p + 1 - |\tilde{E}(\mathbb{F}_p)|,$$

$$a_p(E) = a_p(E)a_{p-1}(E) - 1_E(p)p_{a_p-2}(E), \quad e \geq 2,$$

$$a_{mn}(E) = a_m(E)a_n(E), \quad (m, n) = 1,$$

Chapter 9

• Page 368, line 16: Change “ramify in $\mathbb{Q}(\mu_N)$.” to “ramify in $\mathbb{Q}(\mu_N)$ (except that 2 does not ramify if $N \equiv 2 \pmod{4}$, but then $\mathbb{Q}(\mu_N) = \mathbb{Q}(\mu_{N/2})$ and $N/2$ is odd).”

• Page 375, line 5: Change “$1 + \ell^n\mathbb{Z}_\ell$” to “$1 + \ell^n\mathbb{Z}_\ell$”.

• Page 383, middle of the page: Replace “For each $n$ the field $\mathbb{Q}(E[\ell^n])$ is a Galois number field, giving a restriction map

$$G_{\mathbb{Q}} \rightarrow \text{Gal}(\mathbb{Q}(E[\ell^n])/\mathbb{Q}), \quad \sigma \mapsto \sigma|_{\mathbb{Q}(E[\ell^n])},$$

as
and there is also an injection
\[ \text{Gal}(\mathbb{Q}(E[\ell^n])/\mathbb{Q}) \to \text{Aut}(E[\ell^n]) \]
with “Under the isomorphic identification of \( E \) with \( \text{Pic}^0(E) \), multiplication by \( \ell^n \) on \( E \) for any \( n \in \mathbb{Z}^+ \) becomes purely formal on \( \text{Pic}^0(E) \), and so it clearly commutes with the \( G_\mathbb{Q} \)-action on \( \text{Pic}^0(E) \). Thus the actions on \( E \) commute as well, and so the Galois action restricts to \( \ell^n \)-torsion,
\[ G_\mathbb{Q} \to \text{Aut}(E[\ell^n]) \]."

- Page 384, line (-10): Replace “Theorem 8.4.4” by “Proposition 8.4.4”.
- Page 384: The second paragraph of the proof of Theorem 9.4.1 can be improved as follows.
  “For the characteristic polynomial, consider the diagram
  \[
  \begin{array}{ccc}
  E[\ell^n] & \xrightarrow{a_p(E)} & E[\ell^n] \\
  \downarrow & & \downarrow \\
  \tilde{E}[\ell^n] & \xrightarrow{\sigma_p + p\sigma_p^{-1}} & \tilde{E}[\ell^n].
  \end{array}
  \]
  By Proposition 8.3.2, \( a_p(E) \) acts on \( \text{Pic}^0(\tilde{E}) \) as multiplication by \( \sigma_p \cdot \sigma_p^* + \sigma_p^* \).
  Thus, identifying elliptic curves with their degree-0 Picard groups as earlier, and recalling from equations (8.14) and (8.15) that \( \sigma_p = \sigma_p \cdot \sigma_p^* \) and \( p\sigma_p^{-1} = \sigma_p^* \) under the identification, we see that the diagram commutes. The same diagram but instead with \( \text{Frob}_p + p\text{Frob}_p^{-1} \) across the top row also commutes. Since the vertical arrows are isomorphisms, \( a_p(E) = \text{Frob}_p + p\text{Frob}_p^{-1} \) on \( E[\ell^n] \), and since \( n \) is arbitrary, the equality extends to \( \tau_\ell(E) \). Multiply the equality through by \( \text{Frob}_p \) to get \( \text{Frob}_p^2 - a_p(E)\text{Frob}_p + p = 0 \). Since \( V_\ell(E) \) is 2-dimensional, the characteristic equation of \( \text{Frob}_p \) is as claimed.”
- Page 385: Exercise 9.4.2 requires many changes.
  The exercise applies to the \( t_{p^e}(E) \) rather than to the \( a_{p^e}(E) \), and so this change should be made throughout.
  Change the initial value from \( t_1(E) = 1 \) to \( t_1(E) = 2 \).
  At the end of the text leading up to part (a), delete “except when \( p = 2 \) and \( 2 \mid N \).”
  At the end of part (a), add the sentence, “Note that the equality holds for \( e = 0 \) as well.”
  In part (b), change “Show that” to “Show that for \( e \geq 2 \).”
  Change part (c) to “(c) For \( p \mid N \), (8.11) says that we may take \( \tilde{E} : (y - m_1x)(y - m_2x) = x^3 \) with \( m_1 + m_2, m_1m_2 \in \mathbb{F}_p \). Show that the formula..."
describes a map from \( \mathbb{P}^1(\mathbb{F}_q) \) to \( \tilde{E}(\mathbb{F}_q) \). By considering the map \((x, y) \mapsto y/x \) from \( \tilde{E}(\mathbb{F}_q) - \{(0, 0)\} \) to \( \mathbb{P}^1(\mathbb{F}_q) \), also, show that the displayed map injects except for possibly hitting \((0, 0)\) more than once (when \( m_1, m_2 \) are distinct and lie in \( \mathbb{F}_q \)) and that the map surjects except for possibly missing \((0, 0)\) (when \( m_1, m_2 \) do not lie in \( \mathbb{F}_q \)), and so the map bijects when \( m_1 = m_2 \) lies in \( \mathbb{F}_q \).

The reduction \( \tilde{E} \) is multiplicative if \( m_1 \neq m_2 \). Show that if the reduction is split, i.e., \( m_1, m_2 \in \mathbb{F}_p \), then \( t_{p^e}(E) = 1 \) for all \( e \geq 1 \). Show that if the reduction is non-split, i.e., \( m_1, m_2 \notin \mathbb{F}_p \), then \( t_{p^e}(E) = (-1)^e \) for all \( e \geq 1 \). Show that the recurrence is satisfied in both cases.

The reduction is additive if \( m_1 = m_2 \). Show that the common value \( m \) lies in \( \mathbb{F}_p \), the argument will be different for \( p = 2 \). Show that \( t_{p^e}(E) = 0 \) for all \( e \geq 1 \), and show that the recurrence is satisfied in this case as well.”

Add a new part to the exercise: “(d) Again assume that \( p \nmid N \). Show that

\[
(1 - a_p(E)x + px^2)^{-1} = (1 - \lambda_1x)^{-1}(1 - \lambda_2x)^{-1} = \sum_{c=0}^{\infty} \left( \sum_{c+d=e} \lambda_1^c \lambda_2^d \right) x^e.
\]

Explain why it follows that whereas the normalized prime-power solution-counts of the elliptic curve are \( t_{p^e}(E) = \lambda_1^e + \lambda_2^e \), the corresponding prime-power Dirichlet coefficients of \( L(s, E) \) are \( a_{p^e}(E) = \sum_{c+d=e} \lambda_1^c \lambda_2^d \).

- Page 388, line 4: Replace “The field extension \( \mathbb{Q}(\text{Pic}^0(\mathbb{X}_1(N))[\ell^n])/\mathbb{Q} \) is Galois for each \( n \in \mathbb{Z}^+ \)” with “The Galois action commutes with the purely formal action of multiplication by \( \ell^n \) for any \( n \in \mathbb{Z}^+ \).”
- Page 390: The proof of Lemma 9.5.2 can be clarified as follows.

“Multiplication by \( \ell^n \) is surjective on \( I_f \mathbb{J}_1(N) \). Indeed, it is surjective on the complex torus \( \mathbb{J}_1(N) \), and the commutative Hecke algebra \( \mathbb{H}_\mathbb{Z} \) contains both \( I_f \) and \( \ell^n \), so that \( \ell^n I_f \mathbb{J}_1(N) = I_f \ell^n \mathbb{J}_1(N) = I_f \mathbb{J}_1(N) \).

To show the first statement of the lemma, take any \( y \in A_f[\ell^n] \). Then \( y = x + I_f \mathbb{J}_1(N) \) for some \( x \in \mathbb{J}_1(N) \) such that \( \ell^n x \in I_f \mathbb{J}_1(N) \). Thus \( \ell^n x = \ell^n x' \) for some \( x' \in I_f \mathbb{J}_1(N) \) by the previous paragraph. The difference \( x - x' \) lies in \( \mathbb{J}_1(N)[\ell^n] = \text{Pic}^0(\mathbb{X}_1(N))[\ell^n] \) and maps to \( y \) as desired.

The kernel is \( \text{Pic}^0(\mathbb{X}_1(N))[\ell^n] \cap I_f \mathbb{J}_1(N) = (I_f \mathbb{J}_1(N))[\ell^n] \). We claim that the containment

\[
(I_f \text{Pic}^0(\mathbb{X}_1(N))[\ell^n]) \subset (I_f \mathbb{J}_1(N))[\ell^n],
\]

is in fact equality. Granting the equality, the second statement of the lemma follows quickly: the kernel is now \( (I_f \text{Pic}^0(\mathbb{X}_1(N))[\ell^n]) \). That is, the kernel is \( \text{Pic}^0(\mathbb{X}_1(N))[\ell^n] \cap I_f \text{Pic}^0(\mathbb{X}_1(N)) \), which is stable under the Galois action: the first intersectand is stable because the Galois action on \( \text{Pic}^0(\mathbb{X}_1(N)) \) preserves \( \ell^n \)-torsion, and the second is stable because the Galois and Hecke actions on \( \text{Pic}^0(\mathbb{X}_1(N)) \) commute.
To prove that the containment is equality, note that it is a containment of torsion of $I_f$-images, while if instead we were considering $I_f$-images of torsion then there would be nothing to show, i.e., $\text{Pic}^0(X_1(N))[\ell^n] = J_1(N)[\ell^n]$ and thus $I_f(\text{Pic}^0(X_1(N))[\ell^n]) = I_f(J_1(N)[\ell^n])$. So the argument will relate the given containment of torsion of $I_f$-images to an equality of $I_f$-images of torsion. To do so, let $S_2 = S_2(I_1(N))$ and $H_1 = H_1(X_1(N)[\mathbb{C}; \mathbb{Z}) \subset S_2^\Delta$. Thus $J_1(N) = S_2^\Delta/H_1$ and

$$I_fJ_1(N) = (I_fS_2^\Delta + H_1)/H_1 \cong I_fS_2^\Delta/(H_1 \cap I_fS_2^\Delta).$$

Now suppose that $y \in (I_fJ_1(N))[\ell^n]$. Then $y = x + H_1$ where by the previous display we may take

$$x \in I_fS_2^\Delta \quad \text{and} \quad \ell^n x \in H_1 \cap I_fS_2^\Delta.$$

Proposition 6.2.4 shows that $H_1 \cap I_fS_2^\Delta$ contains $I_fH_1$ as a subgroup of some finite index $M$. Consequently $H_1 \cap I_fS_2^\Delta \subset I_fM^{-1}H_1$. From the previous display and the containment, $\ell^n x \in I_fM^{-1}H_1$, and so

$$x \in I_fM^{-1}x^{-n}H_1.$$

That is, $x = Tx_0$ where $T \in I_f$ and $x_0 \in S_2^\Delta$ and $M\ell^n x_0 \in H_1$, and so $y = T(x_0 + H_1)$ where $x_0 + H_1 \in J_1(N)$ and $M\ell^n(x_0 + H_1) = 0$. In sum, our $y$ from $(I_fJ_1(N))[\ell^n]$ lies in $I_f(J_1(N)[M\ell^n])$, and we are set up to use the equality of $I_f$-images of torsion,

$$y \in I_f(J_1(N)[M\ell^n]) = I_f(\text{Pic}^0(X_1(N))[M\ell^n]) \subset I_f\text{Pic}^0(X_1(N)).$$

And since $\ell^n y = 0$ in fact $y \in (I_f\text{Pic}^0(X_1(N)))[\ell^n]$. Thus the opposite containment is proved, establishing the desired equality. As explained above, the proof of the lemma is complete.”

- Page 391: Replace the two lines before Lemma 9.5.3 with “Since the Tate module $\text{Tat}_\ell(A_f) \cong \mathbb{Z}_{\ell}^{2d}$ is a module over $\mathcal{O}_f$, the tensor product

$$V_\ell(A_f) = \text{Tat}_\ell(A_f) \otimes \mathbb{Q} \cong \mathbb{Q}_{\ell}^{2d}$$

is a module over $\mathcal{O}_f \otimes \mathbb{Q} = \mathbb{K}_f$. Also, it is a module over $\mathbb{Q}_\ell$, with the two actions commuting and with the restrictions of the two actions to $\mathbb{Q}$ agreeing. Thus $V_\ell(A_f)$ is a module over $\mathbb{K}_f \otimes \mathbb{Q} \mathbb{Q}_\ell$.”

Replace the proof of the lemma with “Since $\text{Tat}_\ell(A_f)$ is the inverse limit of the torsion groups $A_f[\ell^n]$, we need to describe $A_f[\ell^n]$ in a fashion that will help establish the freeness. As above, let $S_2 = S_2(I_1(N))$ and let $H_1 = H_1(X_1(N)[\mathbb{C}; \mathbb{Z}) \subset S_2^\Delta$. Consider the quotients $\overline{S_2^\Delta} = S_2^\Delta/I_fS_2^\Delta$ and $\overline{H_1} = (H_1 + I_fS_2^\Delta)/I_fS_2^\Delta$, both $\mathcal{O}_f$-modules. Compute that

$$A_f = J_1(N)/I_fJ_1(N) = (S_2^\Delta/H_1) / ((I_fS_2^\Delta + H_1)/H_1)$$

$$\cong S_2^\Delta/(I_fS_2^\Delta + H_1)$$

$$\cong (S_2^\Delta/I_fS_2^\Delta) / ((H_1 + I_fS_2^\Delta)/I_fS_2^\Delta) = \overline{S_2^\Delta}/\overline{H_1}. $$
Thus \( A_f[\ell^n] \cong \ell^{-n}H_1/H_1 \) for any \( n \in \mathbb{Z}^+ \). The \( \mathcal{O}_f \)-linear isomorphisms \( \ell^{-n}H_1/H_1 \to H_1/\ell^nH_1 \) induced by multiplication by \( \ell^n \) on \( \ell^{-n}H_1 \) assemble to give an isomorphism of \( \mathcal{O}_f \otimes \mathbb{Z}_\ell \)-modules,

\[
\text{Ta}_\ell(A_f) = \varprojlim_n \{ A_f[\ell^n] \} \cong \varprojlim_n \{ \ell^{-n}H_1/H_1 \} \cong \varprojlim_n \{ H_1/\ell^nH_1 \} \cong H_1 \otimes \mathbb{Z}_\ell,
\]

where the transition maps in the last inverse limit are the natural projection maps.

The fact that \( A_f \) is a complex torus of dimension \( d \) and the calculation a moment ago that \( A_f \cong S^\vee_2 / H_1 \) combine to show that the \( \mathcal{O}_f \)-module \( H_1 \cong H_1/(H_1 \cap I_fS^\vee_2) \) has \( \mathbb{Z} \)-rank \( 2d \). Since \( K_f \) is a field, \( H_1 \otimes \mathbb{Q} \) is a free \( K_f \)-module whose \( \mathbb{Q} \)-rank is \( 2d \) and whose \( K_f \)-rank is therefore 2. Consequently, \( H_1 \otimes \mathbb{Q}_\ell = H_1 \otimes \mathbb{Q} \otimes \mathbb{Q}_\ell \) is free of rank 2 over \( K_f \otimes \mathbb{Q}_\ell \). So finally,

\[
V_\ell(A_f) = \text{Ta}_\ell(A_f) \otimes \mathbb{Q} \cong H_1 \otimes \mathbb{Z}_\ell \otimes \mathbb{Q} \cong H_1 \otimes \mathbb{Q}_\ell
\]

is an isomorphism of \( K_f \otimes \mathbb{Q} \mathbb{Q}_\ell \)-modules, and the proof is complete.”

**Hints and Answers to the Exercises**

- Page 396, line (−11): Change “\( \lambda \in \mathcal{O}_{K_f} \)” to “\( \lambda \subseteq \mathcal{O}_{K_f} \)”.
- Page 397, line 11: For the Fermat equation, it is understood that \( \ell \) is an odd prime.

**Hints and Answers to the Exercises**

- Page 414, hint to Exercise 7.7.1: Remove “\( j_N = j(E_j/\langle Q_\tau \rangle) \) and since”, remove “\( j_N^* = j(E_j/\langle Q_\tau^* \rangle) \) and”, and change “In both cases the” to “The”.

**Hints and Answers to the Exercises**

- Page 414, hint to Exercise 7.7.1: Remove “\( j_N = j(E_j/\langle Q_\tau \rangle) \) and since”, remove “\( j_N^* = j(E_j/\langle Q_\tau^* \rangle) \) and”, and change “In both cases the” to “The”.