COMPACTNESS OF $\mathbf{Z}_p$

Let p be prime. The product

$$P = \prod_{m=1}^{\infty} (\mathbf{Z}/p^m\mathbf{Z})$$

is compact by the Tychonoff Theorem since each $\mathbf{Z}/p^m\mathbf{Z}$ is compact. Its elements are sequences,

$$P = \{(x_m + p^m\mathbf{Z})_{m=1}^{\infty}\}.$$

For each positive integer n, the nth projection function,

$$\pi_{n+1,n} : \mathbf{Z}/p^{n+1}\mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}, \qquad x + p^{n+1}\mathbf{Z} \longmapsto x + p^n\mathbf{Z},$$

is continuous (trivially, since its domain and codomain are discrete). Therefore its graph,

$$G_n = \{(x_{n+1} + p^{n+1}\mathbf{Z}, x_n + p^n\mathbf{Z}) : x_n = x_{n+1} \pmod{p^n}\},$$

is a closed subset of $\mathbf{Z}/p^{n+1}\mathbf{Z} \times \mathbf{Z}/p^n\mathbf{Z}$. The corresponding subset of the product P,

$$C_n = G_n \times \prod_{m \neq n, n+1} \mathbf{Z}/p^m\mathbf{Z},$$

is closed as well, because its complement is the open set $G_n^c \times \prod_{m \neq n,n+1} \mathbf{Z}/p^m\mathbf{Z}$.

The p-adic integers form a subspace $\mathbf{Z}_p$ of the product P. Its elements are the *compatible* sequences,

$$\mathbf{Z}_p = \{(x_m + p^m\mathbf{Z})_{m=1}^{\infty} : x_m = x_{m+1} \pmod{p^m} \text{ for } 1 \leq m < \infty\}.$$

That is,

$$\mathbf{Z}_p = \bigcap_{n=1}^{\infty} C_n.$$

Thus $\mathbf{Z}_p$ is closed in the compact product P. Consequently, $\mathbf{Z}_p$ is compact.