THE CYCLOTOMIC ZETA FUNCTION

1. Dirichlet Characters

Let N be a positive integer. A *Dirichlet character modulo N* is defined *initially* as a homomorphism

$$\chi : (\mathbf{Z}/N\mathbf{Z})^\times \longrightarrow \mathbf{C}^\times.$$

Any such character determines a least positive divisor M of N such that the character factors as

$$\chi = \chi_0 \cdot \pi_M : (\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\pi_M} (\mathbf{Z}/M\mathbf{Z})^\times \xrightarrow{\chi_0} \mathbf{C}^\times.$$

The integer M is the *conductor* of χ, and the character χ_0 is *primitive.* Note that if $n \in \mathbf{Z}$ is not coprime to N but is coprime to M then $\chi_0(n + M\mathbf{Z})$ is defined and nonzero. Perhaps confusingly at first, we also use the symbol χ to denote χ_0 lifted to a multiplicative function on the integers,

$$\chi : \mathbf{Z} \longrightarrow \mathbf{C}, \quad \chi(n) = \begin{cases} \chi_0(n + M\mathbf{Z}) & \text{if } \gcd(n, M) = 1, \\ 0 & \text{if } \gcd(n, M) > 1. \end{cases}$$

Thus (the lifted) $\chi(n)$ need not equal (the original) $\chi(n + N\mathbf{Z})$, and in particular $\chi(n)$ need not equal 0 even when $\gcd(n, N) > 1$.

In particular, if $N > 1$ then the trivial character $\mathbf{1}$ modulo N is not identically 1, but it has conductor $M = 1$, and the trivial character $\mathbf{1}_0$ modulo 1 is identically 1 on $(\mathbf{Z}/1\mathbf{Z})^\times = \{\overline{0}\}$, and this character lifts to the constant function $\mathbf{1}(n) = 1$ for all $n \in \mathbf{Z}$.

For any rational prime p, let $p^d \parallel N$ and let $N_p = N/p^d$. (Thus $d = 0$ and $N_p = N$ for all but finitely many p.) Let f denote the order of $p + N_p\mathbf{Z}$ in $(\mathbf{Z}/N_p\mathbf{Z})^\times$, and let $\zeta_f = e^{2\pi i/f}$. For each $k \in \{0, \cdots, f-1\}$, there exist Dirichet characters modulo N that take p to ζ_f^k. The number g of such characters is $g = \phi(N_p)/f$, independently of k.

2. Cyclotomic Arithmetic

Again let N be a positive integer. Let $\zeta_N = e^{2\pi i/N}$, and consider the cyclotomic number field

$$K = \mathbf{Q}(\zeta_N).$$

For any rational prime p, again let $p^d \parallel N$ and let $N_p = N/p^d$. Then p factors in $\mathcal{O}_K$ as

$$p\mathcal{O}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^e,$$

where all the primes lying over p have

- ramification degree $e = \phi(p^d) = p^{d-1}(p-1)$,
- inertial degree $f =$ the order of $p + N_p\mathbf{Z}$ in $(\mathbf{Z}/N_p\mathbf{Z})^\times$,
- and decomposition index $g = \phi(N_p)/f = \phi(N)/(ef)$,
- so that altogether $efg = \phi(N)$.

That is, p ramifies totally in the subfield $\mathbf{Q}(\zeta_{p^d})$ of K, and then it ramifies no further; and the inertial degree f and the decomposition index g for p are as they are in the subfield $\mathbf{Q}(\zeta_{N_p})$ of K. Here it is understood that $\zeta_{p^d} = e^{2\pi i/p^d}$, and similarly for ζ_{N_p}. The rational primes that ramify are precisely the prime divisors of N, except that 2 does not ramify if $2 \mid N$ but $4 \nmid N$. In the exceptional case $N = 2 \pmod 4$, in fact $\mathbf{Q}(\zeta_N) = \mathbf{Q}(\zeta_{N/2})$, so if we avoid redundant cyclotomic extensions $\mathbf{Q}(\zeta_N)$ where $N = 2 \pmod 4$ then the primes that ramify are exactly the prime divisors of N.

3. The Dedekind Zeta Function and its Euler Product

The ring of integers of K is

$$\mathcal{O}_K = \mathbf{Z}[\zeta_N].$$

Define the *norm* of a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ to be

$$\mathrm{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|.$$

Thus we tacitly assert without proof that the quotient is finite. We further assert without proof that the norm is strongly multiplicative, and that

$$\mathrm{N}\mathfrak{p} = p^f \quad \text{where } \mathfrak{p} \mid p \text{ and } f \text{ is the inertial degree as above.}$$

Definition 3.1. *The* **Nth cyclotomic Dedekind zeta function** *is*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathrm{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \mathrm{N}\mathfrak{p}^{-s})^{-1}, \quad \mathrm{Re}(s) > 1.$$

The sum is taken over nonzero ideals of $\mathcal{O}_K$, and the product is taken over maximal ideals.

For any p, compute that

$$\prod_{\mathfrak{p}|p} (1 - \mathrm{N}\mathfrak{p}^{-s})^{-1} = (1 - p^{-fs})^{-g} = \prod_{k=0}^{f-1} (1 - \zeta_f^k p^{-s})^{-g} = \prod_{\chi} (1 - \chi(p) p^{-s})^{-1},$$

where the product is taken over all characters χ modulo N, each character understood to be the underlying primitive character extended to a multiplicative function on $\mathbf{Z}$. In sum, then, we have

$$\zeta_K(s) = \prod_p \prod_\chi (1 - \chi(p) p^{-s})^{-1} = \prod_\chi \prod_p (1 - \chi(p) p^{-s})^{-1},$$

which is to say that the Nth cyclotomic Dedekind zeta function factors as the product of all Dirichlet L-functions modulo N,

$$\boxed{\zeta_K(s) = \prod_\chi L(\chi, s).}$$

One consequence of these calculations is as follows. It is known that the function $L(1, s) = \zeta(s)$, which is initially defined only for $\mathrm{Re}(s) > 1$, extends to a meromorphic function on $\{\mathrm{Re}(s) > 0\}$ whose only singularity is a simple pole at $s = 1$, and it is known that $L(\chi, s)$ for $\chi \neq 1$ is analytic on $\{\mathrm{Re}(s) > 0\}$. Thus $\zeta_N(s)$

is meromorphic on $\{\mathrm{Re}(s) > 0\}$ with its only possible pole at $s = 1$. For positive real s, the estimate

$$(1 - p^{-fs})^{-g} = \left(\sum_{n=0}^{\infty} p^{-nfs}\right)^g \geq \sum_{n=0}^{\infty} p^{-nfgs} \geq \sum_{n=0}^{\infty} p^{-n\phi(N)s} = (1 - p^{-\phi(N)s})^{-1}$$

shows that

$$\zeta_N(s) \geq \zeta(\phi(N)s).$$

Thus $\zeta_N(s)$ must have a pole at $s = 1$, because if it didn't then it would converge on $\{\mathrm{Re}(s) > 0\}$, but the displayed inequality shows that it diverges at $s = 1/\phi(N)$. The fact that $\zeta_N(s)$ has a pole at $s = 1$ is the crux of Dirichlet's proof that there are infinitely many primes in any credible arithmetic progression.

The Dedekind zeta function of K is the generating function for a representation number function,

$$\zeta_K(s) = \sum_{n=1}^{\infty} |\{\mathfrak{a} : \mathrm{N}\mathfrak{a} = n\}| \, n^{-s}.$$

On the other hand, let $d = \phi(N)$. Then

$$\prod_{\chi} L(\chi, s) = \sum_{n=1}^{\infty} \left[\sum_{n_1 \cdots n_d = n} \chi_1(n_1) \cdots \chi_d(n_d) \right] n^{-s}.$$

And so

$$\boxed{|\{\mathfrak{a} : \mathrm{N}\mathfrak{a} = n\}| = \sum_{n_1 \cdots n_d = n} \chi_1(n_1) \cdots \chi_d(n_d).}$$

In particular, for any prime p,

$$|\{\mathfrak{p} : \mathrm{N}\mathfrak{p} = p\}| = \sum_{\chi} \chi(p).$$

The previous display is nothing other than the second orthgonality relation,

$$\sum_{\chi} \chi(p) = \begin{cases} \phi(N_p) & \text{if } p \equiv 1 \pmod{N_p}, \\ 0 & \text{otherwise.} \end{cases}$$

That is, the second orthogonality relation is a special case of a cyclotomic counting formula.