A SECOND WEAK CONTINUATION ARGUMENT FOR EULER–RIEMANN ZETA

The Euler–Rieman zeta function,

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1,$$

is readily continued meromorphically to $\{\mathrm{Re}(s) > 0\}$, and the continuation has only a simple pole at $s = 1$. A simple proof of the continuation has been given in the writeup on Dirichlet's Theorem on primes in an arithmetic progression. Here we give a second proof.

For any prime q consider the Dirichlet series

$$f_q(s) = \sum_{n\geq 1} a_n n^{-s}$$

where the sequence of coefficients $\{a_n\}$ consists of $q-1$ times 1, then a single $1-q$, then $q-1$ more times 1, then another $1-q$, and so on,

$$\{a_n\} = \{1, 1, \cdots, 1, 1-q, 1, 1, \cdots, 1, 1-q, 1, 1, \cdots, 1, 1-q, \cdots\}.$$

Thus the sequence of partial sums of the coefficients is

$$\{A_n\} = \{1, 2, \cdots, q-1, 0, 1, 2, \cdots, q-1, 0, 1, 2, \cdots, q-1, 0, \cdots\}.$$

And so summation by parts shows that the Dirichlet series $f_q(s)$ is analytic on $\mathrm{Re}(s) > 0$.

Compute that for $\mathrm{Re}(s) > 1$ (where we have absolute convergence and therefore may rearrange terms freely),

$$f_q(s) = \sum_{n\geq 1} n^{-s} - q \sum_{n\geq 1} (qn)^{-s} = (1-q^{1-s})\zeta(s), \quad \mathrm{Re}(s) > 1.$$

Since $f_q(s)$ is analytic on $\{\mathrm{Re}(s) > 0\}$ and agrees with $(1-q^{1-s})\zeta(s)$ on $\{\mathrm{Re}(s) > 1\}$, it follows that $(1-q^{1-s})\zeta(s)$ continues analytically to $\{\mathrm{Re}(s) > 0\}$. Therefore $\zeta(s)$ continues meromorphically to $\{\mathrm{Re}(s) > 0\}$ with poles possible only where $q^{1-s} = 1$.

The condition $q^{1-s} = 1$ readily works out to $s \in 1 + 2\pi i\mathbf{Z}/\ln q$. Thus the only possible poles of $\zeta(s)$ in $\{\mathrm{Re}(s) > 0\}$ are distributed evenly along the line $\mathrm{Re}(s) = 1$ with spacing $2\pi/\ln q$. However, q is arbitrary, and a small exercise shows that the sets $2\pi\mathbf{Z}/\ln q$ and $2\pi\mathbf{Z}/\ln q'$ meet only at 0 for distinct primes q and q'. Thus the only possible pole of the extended $\zeta(s)$ is at $s = 1$. This completes the proof.

In any case, the continuation of $\zeta(s)$ to $\{\mathrm{Re}(s) > 0\}$ is superceded by Riemann's proof that the related function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1$$

extends meromorphically to all $s \in \mathbf{C}$, and the continuation has only simple poles at $s = 0, 1$, and the continuation satisfies the functional equation

$$\xi(1-s) = \xi(s), \quad s \in \mathbf{C}.$$