1

Transition to the Adele Group

This lecture transfers functions on the complex upper half plane that satisfy classical conditions to functions on a Lie group that satisfy more natural conditions, and then transfers these functions further to functions on an adele group where we can naturally gather them. Enlarging the Lie group domain gathers all the weights together, and then enlarging the adele group domain gathers all the levels. In particular, all of this applies to Hecke operators and to Fourier coefficients.

1.1 Classical Domain

Consider the classical domain

$$\mathcal{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$$

and the classical group

$$G_{\mathbf{Q}}^{+} = \mathrm{GL}_2^{+}(\mathbf{Q}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{M}_2(\mathbf{Q}) : ad - bc > 0 \right\}.$$

The group $G_{\mathbf{Q}}^{+}$ acts on the domain $\mathcal{H}$ via fractional linear transformations,

$$\gamma z = \frac{az+b}{cz+d}.$$

The standard factor of automorphy is

$$j : G_{\mathbf{Q}}^{+} \times \mathcal{H} \longrightarrow \mathbf{C}, \qquad j(\gamma, z) = cz + d.$$

This satisfies the cocycle condition

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z).$$

For any $\gamma \in G_{\mathbf{Q}}^{+}$ and any $\kappa \in \mathbf{Z}$, the weight κ operator associated to γ is

$$[\gamma]_\kappa : \{\text{functions } f \text{ from } \mathcal{H} \text{ to } \mathbf{C}\} \longrightarrow \{\text{functions } f \text{ from } \mathcal{H} \text{ to } \mathbf{C}\},$$

taking each f to $f[\gamma]_\kappa$, where

$$(f[\gamma]_\kappa)(z) = f(\gamma z) j(\gamma, z)^{-\kappa} (\det \gamma)^{\kappa/2}.$$

For each positive integer L, the classical group $G_{\mathbf{Q}}^{+}$ contains the principal congruence subgroup of level L,

$$\varGamma_L = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) : \gamma \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \pmod{L\mathbf{Z}}\}.$$

Let p be prime, and let

$$\varDelta_p(L) = \{\delta \in \mathrm{M}_2(\mathbf{Z}) : \det(\delta) = p,\ \delta \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right] \pmod{L\mathbf{Z}}\}.$$

The Hecke operator $T_{p,L}$ acts on $[\varGamma_L]_\kappa$-invariant functions $f : \mathcal{H} \longrightarrow \mathbf{C}$ as

$$T_{p,L} f = \sum_{\delta \in \varGamma_L \backslash \varDelta_p(L)} f[\delta]_\kappa, \tag{1.1}$$

which is again $[\varGamma_L]_\kappa$-invariant because $\varGamma_L$ normalizes $\varDelta_p(L)$.

Suppose that the function $f : \mathcal{H} \longrightarrow \mathbf{C}$ satisfies

$$f(z + L) = f(z).$$

For example, this holds if f is $[\varGamma_L]_\kappa$-invariant for any κ. Let $z = x + iy$ and think of f as an $L\mathbf{Z}$-periodic function of the variable x with parameter y. For any integer ℓ let

$$\psi_{\ell/L}(x) = e^{2\pi i \ell x/L}, \quad x \in \mathbf{R}.$$

The set of such characters is a complete orthonormal set for $\mathbf{R}/L\mathbf{Z}$, and so the Fourier series of f is

$$f(z) = \sum_{\ell \in \mathbf{Z}} \psi_{\ell/L}(x) c_{\ell/L}(f, y),$$

where the (ℓ/L)th Fourier coefficient is

$$c_{\ell/L}(f, y) = L^{-1} \int_{\mathbf{R}/L\mathbf{Z}} \psi_{-\ell/L}(\xi) f(\xi + iy)\, d\xi.$$

If f is $\mathcal{C}^2$ then the Fourier series of f converges pointwise to f. If $L \mid L'$, so that also f is L'-periodic, then the Fourier series of f at the two levels agree. We will show this as part of a more general calculation in the adele group environment.

1.2 Classical Domain to Lie Group

Next consider the Lie group associated to $G^+_{\mathbf{Q}}$,

$$G^+_{\mathbf{R}} = \mathrm{GL}^+_2(\mathbf{R}).$$

This group acts on the domain $\mathcal{H}$ via fractional linear transformations. Let $z_0 = i$. To transfer functions on $\mathcal{H}$ to functions on $G^+_{\mathbf{R}}$, fix any $\kappa \in \mathbf{Z}$ and define a map

$$\phi_\kappa : \{\text{functions from } \mathcal{H} \text{ to } \mathbf{C}\} \longrightarrow \{\text{functions from } G^+_{\mathbf{R}} \text{ to } \mathbf{C}\}$$

by

$$\Big(f_{\mathcal{H}} : z \mapsto f_{\mathcal{H}}(z)\Big) \longmapsto \Big(f_{\mathbf{R},\kappa} : g \mapsto f_{\mathcal{H}}(gz_0) j(g, z_0)^{-\kappa} (\det g)^{\kappa/2}\Big).$$

The point z_0 is fixed by a compact subgroup of $G^+_{\mathbf{R}}$,

$$K^+_{\mathbf{R}} = \mathrm{SO}(2).$$

The entire fixing subgroup is $\mathbf{R}^+ K^+_{\mathbf{R}}$ but we want the compact group. The cocycle condition and the fact that $K^+_{\mathbf{R}}$ fixes z_0 combine to show that there is a character

$$\chi_\kappa : K^+_{\mathbf{R}} \longrightarrow \mathbf{C}^\times, \quad \chi_\kappa(k) = j(k, z_0)^{-\kappa}.$$

For any function $f_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathbf{C}$ and any $k \in K^+_{\mathbf{R}}$, the corresponding function $\phi_\kappa f = f_{\mathbf{R},\kappa} : G^+_{\mathbf{R}} \longrightarrow \mathbf{C}$ satisfies

$$f_{\mathbf{R},\kappa}(gk) = f_{\mathbf{R},\kappa}(g)\chi_\kappa(k).$$

That is, $f_{\mathbf{R},\kappa}$ is right $K^+_{\mathbf{R}}$-equivariant of type κ. In fact ϕ_κ is a bijection from the functions on $\mathcal{H}$ to the type κ right $K^+_{\mathbf{R}}$-equivariant functions on $G^+_{\mathbf{R}}$. However, since we soon will be considering a larger class of functions in the Lie group environment and then a larger one yet in the adele group environment, this isn't particularly important.

In particular, $[\ ]_\kappa$-invariance on $\mathcal{H}$ transfers to a more natural invariance on $G^+_{\mathbf{R}}$. For any $\gamma \in G^+_{\mathbf{R}}$ define the composition operator

$$\circ\gamma : \{\text{functions from } G^+_{\mathbf{R}} \text{ to } \mathbf{C}\} \longrightarrow \{\text{functions from } G^+_{\mathbf{R}} \text{ to } \mathbf{C}\}$$

by

$$(f \circ \gamma)(g) = f(\gamma g).$$

Then for any $\gamma \in G^+_{\mathbf{Q}}$ and any $\kappa \in \mathbf{Z}$, the following diagram commutes:

$$\begin{array}{ccc}
\{\text{functions from } \mathcal{H} \text{ to } \mathbf{C}\} & \xrightarrow{\phi_\kappa} & \left\{\begin{matrix}\text{type } \kappa \text{ right } K^+_{\mathbf{R}}\text{-equivariant} \\ \text{functions from } G^+_{\mathbf{R}} \text{ to } \mathbf{C}\end{matrix}\right\} \\
\Big\downarrow {\scriptstyle [\gamma]_\kappa} & & \Big\downarrow {\scriptstyle \circ\gamma} \\
\{\text{functions from } \mathcal{H} \text{ to } \mathbf{C}\} & \xrightarrow{\phi_\kappa} & \left\{\begin{matrix}\text{type } \kappa \text{ right } K^+_{\mathbf{R}}\text{-equivariant} \\ \text{functions from } G^+_{\mathbf{R}} \text{ to } \mathbf{C}\end{matrix}\right\}.
\end{array}$$

If $f_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathbf{C}$ is $[\Gamma_L]_\kappa$-invariant then the diagram shows that the corresponding function $f_{\mathbf{R},\kappa} : G^+_{\mathbf{R}} \longrightarrow \mathbf{C}$ is left Γ_L-invariant, i.e., $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma_L$. In sum, ϕ_κ restricts to a bijection for any L,

$$\phi_{\kappa,L} : \left\{ \begin{matrix} [\Gamma_L]_\kappa\text{-invariant} \\ \text{functions on } \mathcal{H} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{left } \Gamma_L\text{-invariant, type } \kappa \text{ right} \\ K^+_{\mathbf{R}}\text{-equivariant functions on } G^+_{\mathbf{R}} \end{matrix} \right\}.$$

The conditions on the right side are considerably tidier, defined on the Lie group $G^+_{\mathbf{R}}$ rather than the domain $\mathcal{H}$, so that now Γ_L is a subgroup (as $L\mathbf{Z}$ is a subgroup of $\mathbf{R}$ in the one-dimensional situation), and separating the awkward condition of $[\ ]_\kappa$-invariance under Γ_L as a group that acts on $\mathcal{H}$ into the natural conditions of left invariance under Γ_L and type κ right equivariance under $K^+_{\mathbf{R}}$ as subgroups of $G^+_{\mathbf{R}}$.

The Hecke operator $T_{p,L}$ on left Γ_L-invariant functions $f_{\mathbf{R}} : G^+_{\mathbf{R}} \longrightarrow \mathbf{C}$ is defined to make the following diagram commute:

$$\begin{array}{ccc} f_{\mathcal{H}} & \xrightarrow{\phi_{\kappa,L}} & f_{\mathbf{R}} \\ \downarrow T_{p,L} & & \downarrow T_{p,L} \\ T_{p,L} f_{\mathcal{H}} & \xrightarrow{\phi_{\kappa,L}} & T_{p,L} f_{\mathbf{R}}. \end{array}$$

That is, $T_{p,L} f_{\mathbf{R}} = (T_{p,L} f_{\mathcal{H}})_{\mathbf{R}}$. For any $\gamma \in G^+_{\mathbf{Q}}$ and $g \in G^+_{\mathbf{R}}$, since $(f_{\mathcal{H}}[\gamma]_\kappa)_{\mathbf{R}}(g) = f_{\mathbf{R}}(\gamma g)$ by the earlier commutative diagram, it follows from (1.1) that the Lie group Hecke operator is

$$T_{p,L} f_{\mathbf{R}} = \sum_{\delta \in \Gamma_L \backslash \Delta_p(L)} f_{\mathbf{R}} \circ \delta. \tag{1.2}$$

We next enlarge the Lie group environment. Rather than continue keeping track of each weight κ separately, consider left Γ_L-invariant functions on $G^+_{\mathbf{R}}$, or equivalently, functions on the quotient space $\Gamma_L \backslash G^+_{\mathbf{R}}$. For each κ the map $\phi_{\kappa,L}$ can be viewed as is an injection into this κ-independent space,

$$\phi_{\kappa,L} : \{[\Gamma_L]_\kappa\text{-invariant functions on } \mathcal{H}\} \longrightarrow \mathcal{C}(\Gamma_L \backslash G^+_{\mathbf{R}}).$$

In particular, the Lie group definition of $T_{p,L}$ in (1.2) makes no reference to a weight or to right equivariance.

Now we consider Fourier series on $G^+_{\mathbf{R}}$. This is facilitated by the *Iwasawa decomposition*,

$$G^+_{\mathbf{R}} = N_{\mathbf{R}} M^+_{\mathbf{R}} K^+_{\mathbf{R}}$$

where the first two subgroups are

$$N_{\mathbf{R}} = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbf{R} \right\}, \quad M^+_{\mathbf{R}} = \left\{ \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} : y_1, y_2 \in \mathbf{R}^+ \right\},$$

and as before, $K^+_{\mathbf{R}} = \mathrm{SO}(2)$. Indeed, recall that $z_0 = i$, and for any $g \in G^+_{\mathbf{R}}$ let $gz_0 = z = x + iy$. Then also $z = n_x m_y z_0$ where

$$n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad m_y = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}.$$

So $(n_x m_y)^{-1} g$ takes the form $rk \in \mathbf{R}^+ K^+_{\mathbf{R}}$, and thus $g = n_x \cdot r m_y \cdot k$ as desired. It is easy to see that the decomposition is unique.

Suppose that the function $f : G^+_{\mathbf{R}} \longrightarrow \mathbf{C}$ satisfies

$$f\left(\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} g\right) = f(g).$$

Consider the group

$$N_{\mathbf{Z},L} = \left\{\begin{bmatrix} 1 & L\mathbf{Z} \\ 0 & 1 \end{bmatrix}\right\} \subset N_{\mathbf{R}}.$$

Let $g = nmk$ and think of f as a left $N_{\mathbf{Z},L}$-invariant function of the variable n with parameter mk. For any integer ℓ, let

$$\psi_{\ell/L}(n_x) = e^{2\pi i \ell x/L}, \quad n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N_{\mathbf{R}}.$$

The Fourier series of f is

$$f(g) = \sum_{\ell \in \mathbf{Z}} \psi_{\ell/L}(n) c_{\ell/L}(f, mk),$$

where the (ℓ/L)th Fourier coefficient is

$$c_{\ell/L}(f, mk) = L^{-1} \int_{N_{\mathbf{Z},L} \backslash N_{\mathbf{R}}} \psi_{-\ell/L}(\nu) f(\nu mk)\, d\nu. \tag{1.3}$$

This discussion is set in $\mathcal{C}(N_{\mathbf{Z},L} \backslash G^+_{\mathbf{R}})$, a larger space than $\mathcal{C}(\Gamma_L \backslash G^+_{\mathbf{R}})$ earlier in the section. For any weight κ the $[\Gamma_L]_\kappa$-invariant functions on $\mathcal{H}$ are $L\mathbf{Z}$-periodic, and there is a commutative diagram of injections

$$\begin{array}{ccc} \left\{\begin{matrix} [\Gamma_L]_\kappa\text{-invariant} \\ \text{functions on } \mathcal{H} \end{matrix}\right\} & \xrightarrow{\phi_{\kappa,L}} & \mathcal{C}(\Gamma_L \backslash G^+_{\mathbf{R}}) \\ \downarrow & & \downarrow \\ \left\{\begin{matrix} L\mathbf{Z}\text{-periodic} \\ \text{functions on } \mathcal{H} \end{matrix}\right\} & \longrightarrow & \mathcal{C}(N_{\mathbf{Z},L} \backslash G^+_{\mathbf{R}}), \end{array}$$

but the map across the bottom row depends on κ. That is, no map from the $L\mathbf{Z}$-periodic functions on $\mathcal{H}$ to $\mathcal{C}(N_{\mathbf{Z},L} \backslash G^+_{\mathbf{R}})$ is compatible with different weights. The simplest, most natural map across the bottom is the one compatible with weight 0.

1.3 Lie Group to Adele Group

Introduce the global notation

$$G = \mathrm{GL}_2,$$

so that $G_{\mathbf{Q}}$ denotes $\mathrm{GL}_2(\mathbf{Q})$ and $G_{\mathbf{R}}$ denotes $\mathrm{GL}_2(\mathbf{R})$. Similarly the associated adele group is

$$G_{\mathbf{A}} = \mathrm{GL}_2(\mathbf{A}).$$

This group contains a copy of $G_{\mathbf{Q}}$ embedded at each component and therefore a copy of $G_{\mathbf{Q}}$ embedded diagonally. The group $G_{\mathbf{Q}}$ is identified with its diagonally embedded image in $G_{\mathbf{A}}$, and the group $G_{\mathbf{R}}$ is naturally identified with the infinite component of $G_{\mathbf{A}}$. The group $G_{\mathbf{A}}$ contains a compact subgroup associated to any positive integer L, as follows. For each finite place v of $\mathbf{Q}$, the local group is

$$G_v = \mathrm{GL}_2(\mathbf{Q}_v).$$

Define a compact subgroup of the local group,

$$K_v = \mathrm{GL}_2(\mathbf{Z}_v),$$

and define an L-dependent compact subgroup of the local group as well,

$$K_{v,L} = \{g \in K_v : g \equiv 1 \pmod{L\mathbf{Z}_v}\}.$$

(Here 1 denotes the 2×2 identity matrix.) Thus $K_{v,L} = K_v$ if $v \nmid L$. The product

$$K_{\mathbf{f},L} = \prod_{v<\infty} K_{v,L}$$

is the finite global compact subgroup of $G_{\mathbf{A}}$ associated to L. Recall the group $K_{\mathbf{R}}^+ = \mathrm{SO}(2)$. This group embeds in $G_{\mathbf{A}}$ at the infinite place, and so it makes sense to enlarge the subgroup $K_{\mathbf{f},L}$ of $G_{\mathbf{A}}$ to

$$K_{\mathbf{A},L}^+ = K_{\mathbf{f},L} K_{\mathbf{R}}^+.$$

The same applies to the group $K_{\mathbf{R}} = \mathrm{O}(2)$, giving

$$K_{\mathbf{A},L} = K_{\mathbf{f},L} K_{\mathbf{R}}.$$

Proposition 1.3.1. *The embedding of $G_{\mathbf{R}}^+$ in $G_{\mathbf{A}}$ induces a map*

$$\iota_L : \Gamma_L \backslash G_{\mathbf{R}}^+ \longrightarrow G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}, \quad \Gamma_L g \longmapsto G_{\mathbf{Q}} g K_{\mathbf{f},L}. \tag{1.4}$$

This map is an injection.

Proof. To see that ι_L is well defined, replace g by γg for any $\gamma \in \Gamma_L$. Let $\gamma_{\mathbf{f}}\gamma_{\mathbf{R}}$ be the diagonally embedded image of γ in $G_{\mathbf{A}}$. Compute that

$$\begin{aligned} G_{\mathbf{Q}}\gamma_{\mathbf{R}} g K_{\mathbf{f},L} &= G_{\mathbf{Q}}\gamma^{-1}\gamma_{\mathbf{R}} g \gamma_{\mathbf{f}} K_{\mathbf{f},L} \quad \text{since } \gamma^{-1} \in G_{\mathbf{Q}} \text{ and } \gamma_{\mathbf{f}} \in K_{\mathbf{f},L} \\ &= G_{\mathbf{Q}} g K_{\mathbf{f},L}. \end{aligned}$$

To see that ι_L is injective, suppose that $\iota_L(\Gamma_L g') = \iota_L(\Gamma_L g)$, i.e.,

$$g' = \gamma g k, \quad g', g \in G^+_{\mathbf{R}},\ \gamma \in G_{\mathbf{Q}},\ k \in K_{\mathbf{f},L}.$$

Then $\gamma_{\mathbf{f}} = k_{\mathbf{f}}^{-1}$, and so $\gamma \in \mathrm{GL}_2(\mathbf{Z})$ and $\gamma \equiv 1 \pmod L$. The relation at the ∞-place (i.e., in $G^+_{\mathbf{R}}$) is $g' = \gamma g$, so that $\gamma \in \Gamma_L$ and thus $\Gamma_L g' = \Gamma_L g$. □

The fact that ι_L is an injection automatically gives a surjection

$$\pi_L : \mathcal{C}(G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/K_{\mathbf{f},L}) \longrightarrow \mathcal{C}(\Gamma_L\backslash G^+_{\mathbf{R}}).$$

Letting suitably invariant functions act on elements rather than on cosets, the inverse image of a function $f_{\mathbf{R}} : G^+_{\mathbf{R}} \longrightarrow \mathbf{C}$ is the set of all functions $f_{\mathbf{A}} : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ satisfying

$$f_{\mathbf{A}}(g_{\mathbf{R}}) = f_{\mathbf{R}}(g_{\mathbf{R}}), \quad g \in G^+_{\mathbf{R}}, \tag{1.5}$$

where the first $g_{\mathbf{R}}$ in the display is the embedded image of the second one. If $f_{\mathbf{R}}$ is type κ right $K^+_{\mathbf{R}}$-equivariant and $f_{\mathbf{A}}(g_{\mathbf{f}} g_\infty) = f_{\mathbf{A}}(g_{\mathbf{f}}) f_{\mathbf{A}}(g_\infty)$ then $f_{\mathbf{A}}$ is type κ right $K^+_{\mathbf{A},L}$-equivariant, meaning

$$f_{\mathbf{A}}(gk) = f_{\mathbf{A}}(g)\chi_\kappa(k_\infty), \quad k \in K^+_{\mathbf{A},L}.$$

But this condition does not hold for all $f_{\mathbf{A}}$ in $\pi_L^{-1} f_{\mathbf{R}}$.

The Hecke operator $T_{p,L}$ on left $G_{\mathbf{Q}}$-invariant, right $K_{\mathbf{f}}$-invariant functions $f_A : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ is defined to make the following diagram to commute:

$$\begin{array}{ccc} f_{\mathbf{A}} & \stackrel{\pi_L}{\longmapsto} & f_{\mathbf{R}} \\ {\scriptstyle T_{p,L}}\downarrow & & \downarrow{\scriptstyle T_{p,L}} \\ T_{p,L} f_{\mathbf{A}} & \stackrel{\pi_L}{\longmapsto} & T_{p,L} f_{\mathbf{R}}. \end{array}$$

So compute, using (1.2) and then (1.5), that going across the top of the diagram and then down the right side gives for any $g_{\mathbf{R}} \in G^+_{\mathbf{R}}$,

$$(T_{p,L}\pi_L f_{\mathbf{A}})(g_{\mathbf{R}}) = \Big(\sum_\delta \pi_L f_{\mathbf{A}} \circ \delta\Big)(g_{\mathbf{R}}) = \sum_\delta (\pi_L f_{\mathbf{A}})(\delta g_{\mathbf{R}}) = \sum_\delta f_{\mathbf{A}}(\delta_{\mathbf{R}} g_{\mathbf{R}}).$$

Since $\delta \in G_{\mathbf{Q}}$ it follows that $G_{\mathbf{Q}}\delta_{\mathbf{R}} = G_{\mathbf{Q}}\delta_{\mathbf{f}}^{-1}$, and so each summand is

$$f_A(\delta_{\mathbf{R}} g_{\mathbf{R}}) = f_A(\delta_{\mathbf{f}}^{-1} g_{\mathbf{R}}) = f_A(g_{\mathbf{R}}\delta_{\mathbf{f}}^{-1}).$$

Let v be the finite place corresponding to p, and identify $\delta_v^{-1} \in G_v$ with its image in $G_{\mathbf{A}}$. Since $\delta_w^{-1} \in K_{w,L}$ for all other finite places w, it follows that $\delta_{\mathbf{f}}^{-1} K_{\mathbf{f},L} = \delta_v^{-1} K_{\mathbf{f},L}$ and thus $f_A(g_{\mathbf{R}} \delta_{\mathbf{f}}^{-1}) = f_A(g_{\mathbf{R}} \delta_v^{-1})$. In sum, the Hecke operator on $\mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L})$ is suitably defined as

$$(T_{p,L} f)(g) = \sum_{\delta \in \Gamma_L \backslash \Delta_p(L)} f(g \delta_v^{-1}), \quad g \in G_{\mathbf{A}}. \tag{1.6}$$

This makes it obvious that $T_{p,L} T_{q,L} = T_{q,L} T_{p,L}$ for distinct primes p and q.

Just as enlarging the Lie group environment gathers all the weights together, enlarging the adele group environment gathers all the levels. Take the (inverse) limit

$$\lim_L (G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}) = G_{\mathbf{Q}} \backslash G_{\mathbf{A}}.$$

(*Email from Paul: Let K be a compact totally disconnected group acting effectively from the right on a topological space X, meaning that $xk \neq x$ unless $k = 1$. Let K_i be a basis at the identity in K. Then $\lim_i (X/K_i) \cong X$.*) In the Lie group environment, $\lim_L (\Gamma_L \backslash G_{\mathbf{R}}^+)$ has no convenient description. A left $G_{\mathbf{Q}}$-invariant function $f : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ that is defined on some quotient $G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}$ is called uniformly locally constant on the right. The transition from the classical domain to the Lie group and then the adele group is summarized in the following diagram, in which $\phi_{\kappa,L}$ and φ_L are injections and π_L is a surjection:

$$\begin{array}{ccc} & & \mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L}) \xrightarrow{\varphi_L} \mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}}) \\ & & \downarrow \pi_L \\ \left\{ \begin{array}{c} [\Gamma_L]_\kappa\text{-invariant} \\ \text{functions on } \mathcal{H} \end{array} \right\} & \xrightarrow{\phi_{\kappa,L}} & \mathcal{C}(\Gamma_L \backslash G_{\mathbf{R}}^+) \end{array}$$

As mentioned, $\mathcal{C}(\Gamma_L \backslash G_{\mathbf{R}}^+)$ incorporates all weights and $\mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$ incorporates all levels. If we had used SL_2 as our group throughout the discussion then the strong approximation theorem would make the vertical arrow a bijection, but we need the larger group for the Hecke operators and for pending calculations with the Fourier coefficients. A left $G_{\mathbf{Q}}$-invariant function $f : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ is called an automorphic form. An automorphic form is called a cusp form if

$$\int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} f(\nu g)\, d\nu = 0, \quad g \in G_{\mathbf{A}}.$$

This will be explained soon.

The level L Hecke operator $T_{p,L}$ on $\mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / K_{\mathbf{f},L})$, defined by (1.6), does not obviously extend to $\mathcal{C}(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$ since the right $K_{\mathbf{f},L}$-invariance of f is required to make the summand well defined. However, $T_{p,L}$ can be rewritten as an integral that does extend. Let v be the finite place corresponding to p. Recall the groups G_v and $K_{v,L}$. Define

$$\Delta_v(L) = \{\delta \in \mathrm{M}_2(\mathbf{Z}_v) : \det\delta \in p\mathbf{Z}_v^\times,\ \delta \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right] \pmod{L\mathbf{Z}_v}\}.$$

Let η_L be the characteristic function of $\Delta_v^{-1}(L)$ on G_v. Since $K_{v,L}$ is compact and open in G_v, the quotient $G_v/K_{v,L}$ is discrete. Also view η_L as defined on $G_v/K_{v,L}$. Let $\mu_L = \mu(K_{v,L})$. If μ_1 is normalized to 1 and $L = \prod_{v<\infty} p_v^{e_v}$ then although the value of μ_L is not particularly important, we can note that

$$\mu_L^{-1} = |K_v/K_{v,L}| = |\mathrm{GL}_2(\mathbf{Z}_v/L\mathbf{Z}_v)| = |\mathrm{GL}_2(\mathbf{Z}/p_v^{e_v}\mathbf{Z})|.$$

For any $f \in \mathcal{C}(G_\mathbf{Q}\backslash G_\mathbf{A}/K_{\mathbf{f},L})$ and any $g \in G_\mathbf{A}$, compute

$$\int_{G_v} f(gh)\eta_L(h)\,dh = \mu_L \sum_{h\in G_v/K_{v,L}} f(gh)\eta_L(h) = \mu_L \sum_{\delta\in K_{v,L}\backslash\Delta_v(L)} f(g\delta^{-1}).$$

The embedding $G_\mathbf{Q} \longrightarrow G_v$ induces a bijection $\Gamma_L\backslash\Delta_p(L) \longrightarrow K_{v,L}\backslash\Delta_v(L)$. (`Check surjectivity.`) Consequently, for any $f \in \mathcal{C}(G_\mathbf{Q}\backslash G_\mathbf{A}/K_{\mathbf{f},L})$ the integral gives the Hecke operator,

$$\int_{G_v} f(gh)\eta_L(h)\,dh = \mu_L \sum_{\delta\in\Gamma_L\backslash\Delta_p(L)} f(g\delta_v^{-1}), \quad g \in G_\mathbf{A}.$$

But the integral makes sense for any $f \in \mathcal{C}(G_\mathbf{Q}\backslash G_\mathbf{Q})$. Therefore it is the definition of the level L Hecke operator $T_{p,L}$ on $\mathcal{C}(G_\mathbf{Q}\backslash G_\mathbf{A})$,

$$(T_{p,L}f)(g) = \mu_L^{-1}\int_{G_v} f(gh)\eta_L(h)\,dh, \quad g \in G_\mathbf{A}.$$

And once we stop thinking about the classical environment, we can drop the artifact-constant μ_L^{-1} from this definition.

More generally, the local Hecke algebra is a convolution algebra. Let $\mathcal{C}_\mathrm{c}^\infty(G_v)$ denote the set of locally constant functions on G_v with compact support. If η_1 and η_2 belong to $\mathcal{C}_\mathrm{c}^\infty(G_v)$ then so does their convolution $\eta_1 * \eta_2$, defined as

$$(\eta_1 * \eta_2)(g) = \int_{G_v} \eta_1(gh^{-1})\eta_2(h)\,dh, \quad g \in G_v.$$

For any $\eta \in \mathcal{C}_\mathrm{c}^\infty(G_v)$ and any integrable function $f : G_v \longrightarrow \mathbf{C}$, define the integral operator

$$(\eta \cdot f)(g) = \int_{G_v} f(gh)\eta(h)\,dh, \quad g \in G_v.$$

Then for all η_1, η_2, and f,

$$(\eta_1 * \eta_2)\cdot f = \eta_1 \cdot (\eta_2 \cdot f).$$

The spherical Hecke algebra (or the Iwahori–Hecke algebra) is the subalgebra of functions η that are left and right K_v-invariant. Gelfand showed that the spherical Hecke algebra is commutative, using the Borel–Matsumoto theorem.

We return to Fourier coefficients. Recall the definition $N_{\mathbf{Z},L} = \{\left[\begin{smallmatrix}1 & L\mathbf{Z}\\ 0 & 1\end{smallmatrix}\right]\}$. There is a diagram similar to the previous one,

$$\begin{array}{ccc} \mathcal{C}(N_{\mathbf{Q}}\backslash G_{\mathbf{A}}/K_{\mathbf{f},L}) & \xrightarrow{\varphi_L} & \mathcal{C}(N_{\mathbf{Q}}\backslash G_{\mathbf{A}}) \\ \Big\downarrow{\scriptstyle \pi_L} & & \\ \mathcal{C}(N_{\mathbf{Z},L}\backslash G_{\mathbf{R}}^+) & & \end{array}$$

Since now the group is GL_2 rather than GL_2^+, the Iwasawa decomposition at the infinite component changes to

$$G_{\mathbf{R}} = N_{\mathbf{R}} M_{\mathbf{R}} K_{\mathbf{R}}$$

where the new subgroups are

$$M_{\mathbf{R}} = \left\{ \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} : y_1, y_2 \in \mathbf{R}^\times \right\}, \quad K_{\mathbf{R}} = \mathrm{O}(2).$$

The Iwasawa decomposition of each finite local component $G_v = \mathrm{GL}_2(\mathbf{Q}_v)$ is

$$G_v = N_v M_v K_v$$

where the first two subgroups are

$$N_v = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbf{Q}_v \right\}, \quad M_v = \left\{ \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} : y_1, y_2 \in \mathbf{Q}_v^\times \right\},$$

and as before, $K_v = \mathrm{GL}_2(\mathbf{Z}_v)$. To see this, compute that for any $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in G_v$, any $n = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \in N_v$, and any $m = \left[\begin{smallmatrix} y_1 & 0 \\ 0 & y_2 \end{smallmatrix}\right] \in M_v$,

$$m^{-1}n^{-1}g = \begin{bmatrix} y_1^{-1} & 0 \\ 0 & y_2^{-1} \end{bmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} y_1^{-1}(a - xc) & y_1^{-1}(b - xd) \\ y_2^{-1}c & y_2^{-1}d \end{bmatrix}.$$

Since c and d can't both be zero, at least one of c/d, d/c is defined and integral. If c/d is integral then set $y_1 = \Delta/d$, $y_2 = d$, and $x = b/d$ to make the right side $\left[\begin{smallmatrix} 1 & 0 \\ c/d & 1 \end{smallmatrix}\right]$. And if d/c is integral then set $y_1 = -\Delta/c$, $y_2 = c$, and $x = a/c$ to make the right side $\left[\begin{smallmatrix} 0 & 1 \\ 1 & d/c \end{smallmatrix}\right]$. The Iwasawa decomposition is no longer unique, but n and mk are still unique at the infinite component.

Suppose that the function $f : G_{\mathbf{A}} \longrightarrow \mathbf{C}$ is left $N_{\mathbf{Q}}$-invariant. Even though the Iwasawa decomposition is not unique, think of f as a function of the variable $n \in N_{\mathbf{A}}$ with parameter $mk \in M_{\mathbf{A}} K_{\mathbf{A}}$. Then the Fourier expansion of f is a sum over characters ψ on $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}$. At least in some L^2 sense,

$$f(g) = \sum_{\psi} \psi(n) c_{\psi}(f, mk)$$

where the Fourier coefficients are

$$c_{\psi}(f, mk) = \int_{N_{\mathbf{Q}}\backslash N_{\mathbf{A}}} \psi^{-1}(\nu) f(\nu mk)\, d\nu. \tag{1.7}$$

The measure here is normalized so that $\mu(N_{\mathbf{Q}}\backslash N_{\mathbf{A}}) = 1$. When ψ is the trivial character, call this the 0th Fourier coefficient or the constant term along N. This explains the definition of cusp form earlier in the section. Since $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}$ is compact, if f is continuous then the integral exists. Also if f is locally constant at each finite place and is $\mathcal{C}^{\infty}$ at the infinite place then the Fourier series converges to f pointwise. Despite the nonunique decomposition, it is straightforward to verify that the Fourier series is well defined, i.e., if $nmk = n'm'k'$ then $\psi(n)c_{\psi}(f, mk) = \psi(n')c_{\psi}(f, m'k')$.

The adele group Fourier series is indeed a generalization from the Lie group. Suppose $f_{\mathbf{A}} \in \mathcal{C}(N_{\mathbf{Q}}\backslash G_{\mathbf{A}})$ takes the form $f_{\mathbf{A}} = \varphi_L f_{\mathbf{A},L}$ for some $f_{\mathbf{A},L} \in \mathcal{C}(N_{\mathbf{Q}}\backslash G_{\mathbf{A}}/K_{\mathbf{f},L})$. Define

$$N_{\mathbf{A},L} = \prod_{v} N_{v,L}, \quad N_{v,L} = \begin{cases} N_v \cap K_{v,L} & v < \infty, \\ N_{\mathbf{R}} & v = \infty. \end{cases}$$

Then $N_{\mathbf{Q}}N_{\mathbf{A},L}\backslash N_{\mathbf{A}} \cong N_{\mathbf{Z},L}\backslash N_{\mathbf{R}}$. Thus for any character ψ of $N_{\mathbf{Q}}\backslash N_{\mathbf{A}}$ and for any element $g_{\mathbf{R}}$ of $G^+_{\mathbf{R}}$ embedded in $G_{\mathbf{A}}$, we may take an Iwasawa decomposition of $g_{\mathbf{R}}$ in $G^+_{\mathbf{R}}$ and then the ψth Fourier coefficient of $f_{\mathbf{A}}(g_{\mathbf{R}})$ works out to

$$c_{\psi}(f_{\mathbf{A}}, m_{\mathbf{R}}k_{\mathbf{R}}) = \int_{N_{\mathbf{Q}}N_{\mathbf{A},L}\backslash N_{\mathbf{A}}} \psi^{-1}(\nu) f_{\mathbf{A},L}(\nu m_{\mathbf{R}}k_{\mathbf{R}})\, d\nu \cdot \int_{N_{\mathbf{Q}}\backslash N_{\mathbf{Q}}N_{\mathbf{A},L}} \psi^{-1}(\kappa)\, d\kappa.$$

The second integral is 0 unless ψ is a character of $N_{\mathbf{Q}}N_{\mathbf{A},L}\backslash N_{\mathbf{A}}$, in which case it is $\mu(N_{\mathbf{Q}}\backslash N_{\mathbf{A}})/\mu(N_{\mathbf{Q}}N_{\mathbf{A},L}\backslash N_{\mathbf{A}}) = L^{-1}$. Let $f_{\mathbf{R}} = \pi_L f_{\mathbf{A},L}$, and view ψ as a character of $N_{\mathbf{Z},L}\backslash N_{\mathbf{R}}$. Then the Fourier coefficient is

$$c_{\psi}(f_{\mathbf{A}}, m_{\mathbf{R}}k_{\mathbf{R}}) = L^{-1} \int_{N_{\mathbf{Z},L}\backslash N_{\mathbf{R}}} \psi^{-1}(\nu) f_{\mathbf{R}}(\nu m_{\mathbf{R}}k_{\mathbf{R}})\, d\nu.$$

Comparing this to (1.3) shows that the adele group Fourier series of $f_{\mathbf{A}}(g_{\mathbf{R}})$ is the Lie group Fourier series of $f_{\mathbf{R}}(g_{\mathbf{R}})$, as claimed.

Let $f : N_{\mathbf{Q}}\backslash G_{\mathbf{A}} \longrightarrow \mathbf{C}$ be left $M_{\mathbf{A}}$-equivariant, meaning that $f(mg) = \chi(m)f(g)$ for some character χ of $M_{\mathbf{A}}$. Such a function gives rise to essentially only one non-0th Fourier coefficient. Since $N_{\mathbf{Q}}\backslash N_{\mathbf{A}} \cong \mathbf{A}/\mathbf{Q}$, again blur the notation by writing $\psi\left(\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]\right) = \psi(x)$. Fix any nontrivial character ψ on $\mathbf{A}/\mathbf{Q}$. Then all other characters take the form

$$\psi_\alpha(x) = \psi(\alpha x), \quad \alpha \in \mathbf{Q}.$$

So ψ_α ranges through the nontrivial characters as α ranges through $\mathbf{Q}^\times$. For any such α, the ψ_αth Fourier coefficient of f on $G_\mathbf{Q}\backslash G_\mathbf{A}$ at g is

$$\begin{aligned} c_{\psi_\alpha}(f, mk) &= \int_{\mathbf{A}/\mathbf{Q}} \psi_\alpha^{-1}(x) f\left(\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] mk\right)\, dx = \int_{\mathbf{A}/\mathbf{Q}} \psi^{-1}(\alpha x) f\left(\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] mk\right)\, dx \\ &= \int_{\mathbf{A}/\mathbf{Q}} \psi^{-1}(x) f\left(\left[\begin{smallmatrix}1 & x/\alpha\\ 0 & 1\end{smallmatrix}\right] mk\right)\, dx. \end{aligned}$$

The last equality here uses the fact that $d(x/\alpha) = dx/|\alpha| = dx$. But $\left[\begin{smallmatrix}1 & x/\alpha\\ 0 & 1\end{smallmatrix}\right] = \left[\begin{smallmatrix}\alpha^{-1} & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right]$ and f is left $M_\mathbf{A}$-equivariant, so this is

$$c_{\psi_\alpha}(f, mk) = \chi\left(\left[\begin{smallmatrix}\alpha^{-1} & 0\\ 0 & 1\end{smallmatrix}\right]\right) c_\psi\left(f, \left[\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right] mk\right).$$

The calculation here makes essential use of GL_2 rather than SL_2.