SKETCH OF THE RIEMANN–VON MANGOLDT EXPLICIT FORMULA

In an online writeup [*add citation when the URL is set*] Paul Garrett says (slight paraphrases in what follows):

> It took 40 years for Hadamard, von Mangoldt, and others to complete Riemann's 1858 sketch of the *Explicit Formula* relating prime numbers to zeros of the Euler–Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does not yield *any* Prime Number Theorem, much less the *optimal* version implied by the Riemman Hypothesis, despite giving a precise relationship between primes and zeros of zeta.
>
> The *idea* of the argument is that the equality of the Euler product and the Riemann–Hadamard product for zeta allows extraction of an *exact formula* for a suitable counting of primes, expressed in terms of a sum over zeros of zeta, via a natural contour integration of the logarithmic derivatives of the products. The difficulties arise in justification of the Hadamard product, in proving that a contour integral can be *threaded between* the zeros of zeta, and that the auxiliary parts of the contour integral go to zero.

This writeup repeats Garrett's encapsulation of the idea of the argument, not addressing the difficulties.

1. Product expressions for the zeta function

The Euler–Riemann zeta function has the product expansion

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1.$$

But also, the completed Euler–Rieman zeta function,

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1,$$

has an integral representation that extends meromorphically to all of $\mathbf{C}$,

$$\xi(s) = \int_{t=1}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t}(t^{s/2} + t^{(1-s)/2})\,\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad s \in \mathbf{C},$$

and the extended ξ visibly satisfies the functional equation

$$\xi(1-s) = \xi(s), \quad s \in \mathbf{C}.$$

The extension of $\xi(s)$ to $\mathbf{C}$ combines with the known extension of $\Gamma(s)$ to $\mathbf{C}$ (via the functional equation $\Gamma(s) = \Gamma(s+1)/s$, for example) to extend $\zeta(s)$ meromorphically to $\mathbf{C}$ as well. Since $\xi(s)$ has simple poles at $s = 0, 1$ and otherwise analytic, and since the gamma function has simple poles at $s = 0, -1, -2, \cdots$, we see that:

> *The extended $\zeta(s)$ has zeros at $s = -2, -4, -6, \cdots$; these are its* **trivial zeros**. *Any other zeros of $\zeta(s)$ lie in the* **critical strip** $0 \le \mathrm{Re}(s) \le 1$.

Assuming that the nontrivial zeros don't bunch up too badly, the entire function $(s-1)\zeta(s)$ has a Weierstrass-type product expansion (in which the symbol ρ denotes nontrivial zeros of $\zeta(s)$),

$$(s-1)\zeta(s) = e^{a+bs} \prod_{n\geq 1} \left(1+\frac{s}{2n}\right) e^{-s/2n} \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho}, \quad s \in \mathbf{C}.$$

Equate the two product expressions for $\zeta(s)$ to get

$$\prod_p (1-p^{-s})^{-1} = \frac{e^{a+bs}}{(s-1)} \prod_{n\geq 1} \left(1+\frac{s}{2n}\right) e^{-s/2n} \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho}, \quad \mathrm{Re}(s) > 1.$$

Take logarithmic derivatives to obtain two expressions for $\zeta'(s)/\zeta(s)$,

$$-\sum_{\substack{p\\ m\geq 1}} \log p \cdot p^{-ms} = b - \frac{1}{s-1} - \sum_{n\geq 1} \frac{s}{2n(s+2n)} + \sum_{\rho} \frac{s}{\rho(s-\rho)}\,, \quad \mathrm{Re}(s) > 1.$$

Since the right side is $\zeta'(s)/\zeta(s)$ for all $s \in \mathbf{C}$, in particular $b+1 = \zeta'(0)/\zeta(0)$, and thus

$$b - \frac{1}{s-1} = b + 1 - \frac{s}{s-1} = \frac{\zeta'(0)}{\zeta(0)} - \frac{s}{s-1}.$$

In sum so far,

$$(1) \quad \sum_{\substack{p\\ m\geq 1}} \log p \cdot p^{-ms} = \frac{s}{s-1} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n\geq 1} \frac{s}{2n(s+2n)} - \sum_{\rho} \frac{s}{\rho(s-\rho)}\,, \quad \mathrm{Re}(s) > 1.$$

2. A useful extraction integral

The plan is to extract number-theoretic information from the previous display (1) via contour integration. To avoid interrupting the story later, we now introduce the identity

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{s}\, ds = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } 0 < x < 1, \end{cases} \qquad \mathrm{Re}(\sigma) > 1.$$

The idea of the proof is that if $x > 1$ then the vertical line of integration slides to the left, picking up a residue at zero, until the integral vanishes, and if $0 < x < 1$ then similarly the line slides to the right, not picking up a residue. A true proof requires an estimate for integrating over truncations of the vertical line, and then estimates of the integrals over the top and bottom of rectangular contours of integration.

3. The Explicit Formula

Conceptually, we will integrate both sides of (1) against the function x^s/s (where $x > 0$ is not a prime power) around an infinite rectangle R; the "rectangle" has right side $\{\mathrm{Re}(s) = \sigma\}$ where $\sigma > 1$, and it extends infinitely high and infinitely far to the left.

Integrating the left side of (1), granting that the integrals over the top, bottom, and left sides of are zero, and then citing the identity in the previous section with

$p^{-m}x$ in place of x, we get

$$\frac{1}{2\pi i}\int_R \sum_{\substack{p\\ m\geq 1}} \log p\cdot p^{-ms}\frac{x^s}{s}\,ds = \sum_{\substack{p\\ m\geq 1}} \frac{\log p}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{(p^{-m}x)^s}{s}\,ds = \sum_{\substack{p\\ m\geq 1\\ p^m<x}} \log p.$$

Meanwhile, the right of (1), slightly rearranged, is

$$\frac{s}{s-1}-\sum_{\rho}\frac{s}{\rho(s-\rho)}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\sum_{n\geq 1}\frac{s}{n(s+2n)}\,.$$

Its extraction integral is evaluated by summing the residues in $\{\mathrm{Re}(s)<\sigma\}$ of each term times x^s/s; e.g., for the last sum, a typical residue-term is $(1/2)x^{-2n}/n$. Equate the right side of the previous extraction calculation and the residue sum to obtain the Explicit Formula: For $x>0$ not a prime power,

$$\boxed{\sum_{\substack{p\\ m\geq 1\\ p^m<x}} \log p = x-\sum_{\rho}\frac{x^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}-\frac{1}{2}\log(1-x^{-2}).}$$

The terms on the right side are arranged by decreasing magnitude. Thus

$$\sum_{p^m<x}\log p\sim x,$$

and the location of the nontrivial zeros of $\zeta(s)$ determines the largest error term. Since the zeros are symmetric about the line $\mathrm{Re}(s)=1/2$, and since $|x^\rho|=x^{\mathrm{Re}(\rho)}$, the error term is as small as possible if the *Riemann hypothesis*—that $\mathrm{Re}(\rho)=1/2$ for all ρ—holds. The constant term $\zeta'(0)/\zeta(0)$ can be shown to equal $\log(2\pi)$.