THE CONDUCTOR OF THE QUADRATIC CHARACTER

1. Completing the Quadratic Reciprocity Law at 2

The characters of $(\mathbf{Z}/8\mathbf{Z})^\times$ are shown in the following table:

	χ_0	χ_1	χ_2	χ_3
1	1	1	1	1
3	1	-1	-1	1
5	1	1	-1	-1
7	1	-1	1	-1

The character group is the $(2,2)$ abelian group, so for example $\chi_1\chi_2 = \chi_3$ and $\chi_i^2 = \chi_0$ for all i. Note that χ_1 can be defined modulo 4, but χ_2 and χ_3 are primitive modulo 8.

Consider two co-prime nonzero integers, the second one positive,

$$\left\{\begin{matrix} P = 2^a P', & P' \text{ odd}, \\ Q = 2^b Q', & Q' > 0 \text{ odd} \end{matrix}\right\} \quad (a, b \in \mathbf{Z}_{\geq 0},\ \min\{a,b\} = 0).$$

We want to establish the general Jacobi symbol reciprocity formula

$$\boxed{(P/Q)\cdot(Q/|P|) = (-1)^{\frac{P'-1}{2}\cdot\frac{Q'-1}{2}}.}$$

The issue is the possible presence of powers of 2 in one of P and Q. As a special case of the desired formula, set $a=0$ and $b=1$ and $Q'=1$ to get

$$(P'/2)\cdot(2/|P'|) = 1.$$

Thus the only possibility is to define $(P'/2) = (2/|P'|)$ for odd P'. To see that this definition does work, and to understand it better, we think with reference to the mod-8 character group. Indeed, two of the characters figure in established reciprocity formulas,

$$(P'/Q')\cdot(Q'/|P'|) = (-1)^{\frac{P'-1}{2}\cdot\frac{Q'-1}{2}},$$
$$(-1/Q') = \chi_1(Q'), \qquad (2/Q') = \chi_2(Q').$$

And naturally the extended Jacobi symbol is to be bimultiplicative. The bimultiplicativity and the second auxiliary formula give

$$(P/Q) = \chi_2(Q')^a (P'/2)^b (P'/Q'), \qquad (Q/|P|) = \chi_2(|P'|)^b (Q'/2)^a (Q'/|P'|),$$

from which the desired formula is

$$\left(\chi_2(|P'|)\cdot(P'/2)\right)^b \cdot \left(\chi_2(Q')\cdot(Q'/2)\right)^a = 1.$$

Thus $(\cdot/2)$ must be $\chi_2 \circ |\ \ |$, and prepending the absolute value has no effect on χ_2. So in sum, we obtain the boxed identity by defining

$$(P'/2) = \chi_2(P'), \quad P' \text{ odd}.$$

And inevitably, $\chi_2(P') = \chi_2(|P'|) = (2/|P'|)$.

2. The Quadratic Character and its Conductor

Let $d \in \mathbf{Z}$ be squarefree, $d \neq 1$. The quadratic field

$$F = \mathbf{Q}(\sqrt{d})$$

has discriminant

$$D_F = \begin{cases} d & \text{if } d = 1 \pmod 4, \\ 4d & \text{if } d = 2, 3 \pmod 4. \end{cases}$$

The corresponding quadratic character of F is

$$\chi_F : \mathbf{Z}_{>0} \longrightarrow \mathbf{C}, \quad \chi_F(n) = (D_F/n).$$

It is not obvious immediately that χ_F is periodic, i.e., that it has a conductor. However, introducing the notation

$$D_F = 2^a \delta, \quad n = 2^b \nu \quad (\delta \text{ and } \nu \text{ odd}),$$

the general Jacobi symbol reciprocity formula gives

$$\chi_F(n) = (-1)^{\frac{\delta-1}{2} \cdot \frac{\nu-1}{2}} \cdot (n/|D_F|) \quad \text{if } \gcd(n, D_F) = 1.$$

Note that

$$\delta = \begin{cases} 1 \pmod 4 & \text{if } d = 1 \pmod 4 \text{ or } d = 2 \pmod 8, \\ 3 \pmod 4 & \text{if } d = 3 \pmod 4 \text{ or } d = 6 \pmod 8. \end{cases}$$

Equivalently, the two cases are $d = 1, 2, 5 \pmod 8$ and $d = 3, 6, 7 \pmod 8$, and in the second case D_F is always even so that $\nu = n$. So now we have

$$\chi_F(n) = \begin{cases} (n/|D_F|) & \text{if } d = 1, 2, 5 \pmod 8, \\ \chi_1(n) \cdot (n/|D_F|) & \text{if } d = 3, 6, 7 \pmod 8 \end{cases} \quad \text{if } \gcd(n, D_F) = 1.$$

More explicitly,

$$\chi_F(n) = \begin{cases} (n/|d|) & \text{if } d = 1 \pmod 4, \\ \chi_1(n) \cdot (n/|d|) & \text{if } d = 3 \pmod 4, \\ \chi_2(n) \cdot (n/|d/2|) & \text{if } d = 2 \pmod 8, \\ \chi_3(n) \cdot (n/|d/2|) & \text{if } d = 6 \pmod 8 \end{cases} \quad \text{if } \gcd(n, D_F) = 1.$$

The respective conductors are

$$\begin{cases} |d| & \text{if } d = 1 \pmod 4, \\ 4|d| & \text{if } d = 3 \pmod 4, \\ 8|d/2| & \text{if } d = 2 \pmod 4, \end{cases}$$

which is to say,

$$\boxed{\chi_F \text{ has conductor } |D_F| \text{ in all cases.}}$$