TAYLOR'S THEOREM FOR POLYNOMIALS

Consider a polynomial

$$f(x) = \sum_{m=0}^{n} c_m x^m.$$

We show algebraically that

$$f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) h^k.$$

First consider a pure monomial function $f_m(x) = x^m$. The binomial expansion

$$(a+h)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} h^k$$

is precisely the Taylor expansion

$$f_m(a+h) = \sum_{k=0}^{m} \frac{1}{k!} f_m^{(k)}(a) h^k.$$

The result for general polynomials follows from this and linearity. Indeed, we have

$$f(x) = \sum_{m=0}^{n} c_m f_m(x),$$

and so by the expansion of $f_m(a+h)$ derived just above,

$$f(a+h) = \sum_{m=0}^{n} c_m f_m(a+h) = \sum_{m=0}^{n} c_m \sum_{k=0}^{m} \frac{1}{k!} f_m^{(k)}(a) h^k.$$

Reverse the order of summation and then note that $f_m^{(k)}(x) = 0$ for $m = 0, \dots, k-1$ to get

$$f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{m=k}^{n} c_m f_m^{(k)}(a) h^k = \sum_{k=0}^{n} \frac{1}{k!} \sum_{m=0}^{n} c_m f_m^{(k)}(a) h^k.$$

Because a linear combination of derivatives is the derivative of the linear combination, this in turn is what we want,

$$f(a+h) = \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{m=0}^{n} c_m f_m \right)^{(k)} (a) h^k = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) h^k.$$

Finally we note that there really are no denominators in these formulas because the binomial coefficients in the expansion of $(a+h)^m$ are integers.