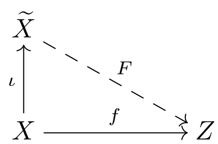
COMPLETING A METRIC SPACE

A *completion* of a metric space X is an isometry from X to a complete metric space,

$$\iota : X \longrightarrow \widetilde{X},$$

which satisfies a mapping property encoded in the diagram



Here the isometry condition is

$$d_{\widetilde{X}}(\iota(x), \iota(x')) = d_X(x, x') \quad \text{for all } x, x' \in X,$$

and the completeness property is that every Cauchy sequence in $\widetilde{X}$ converges in $\widetilde{X}$. Reviewing the Cauchy criterion for a sequence $\{\, \tilde{x}_i \,\}$, for each $\varepsilon > 0$ there exists some starting index i_o such that

$$i, j \geq i_o \implies d_{\widetilde{X}}(\tilde{x}_i, \tilde{x}_j) < \varepsilon.$$

The function $f : X \longrightarrow Z$ in the mapping property is an auxiliary uniformly continuous map to an auxiliary complete metric space, and the mapping property is that there *exists* a *unique* uniformly continuous function $F : \widetilde{X} \longrightarrow Z$ that makes the diagram commute. Reviewing the uniform continuity definition for f, for each $\varepsilon > 0$ there exists some $\delta = \delta(f, \varepsilon) > 0$ such that

$$x, x' \in X, \ d_X(x, x') < \delta \implies d_Z(f(x), f(x')) < \varepsilon.$$

And similarly for F. The existence and essential uniqueness of the completion will be shown below.

Especially, if X is literally a subspace of its completion $\widetilde{X}$, so that the isometry ι is just the inclusion map, then the diagram property is that a uniformly continuous function $f : X \longrightarrow Z$ where Z is a complete metric space extends uniquely to $F : \widetilde{X} \longrightarrow Z$, again uniformly continuous.

The uniform continuity of f, or some property of f that makes it preserve Cauchy sequences, is genuinely required here, rather than just pointwise continuity. To see that uniform continuity preserves Cauchy sequences, let $f : X \longrightarrow Y$ be a uniformly continuous map of metric spaces, and let $\{\, x_i \,\}$ be Cauchy in X. Let $\varepsilon > 0$ be given, and let $\delta = \delta(f, \varepsilon)$. There exists i_o such that $d_X(x_i, x_j) < \delta$ for all $i, j \geq i_o$. Consequently, $d_Y(f(x_i), f(x_j)) < \varepsilon$ for all $i, j \geq i_o$. Thus $\{\, f(x_i) \,\}$ is Cauchy in Y. On the other hand, the pointwise continuous map $f(x) = 1/x$ from $(0, 1]$ to $[1, \infty)$ takes the Cauchy sequence $\{\, 1/n \,\}$ to $\{\, n \,\}$, which is not Cauchy.

An example showing the need for continuous functions that preserve Cauchy sequences in the completion's mapping property is

$$f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

This continuous function has no continuous extension to $\mathbb{R}$. It does not preserve the Cauchy property of a rational sequence that approaches 0 alternatingly above and below, so it cannot be uniformly continuous. One can see the failure of uniform continuity directly as well.

For another example, this time positive, suppose that the metric space X arises from a normed linear space. That is, X is a vector space over the complex number field $\mathbb{C}$, and it carries a norm function

$$|\cdot|_X : X \longrightarrow \mathbb{R}$$

that is positive, absolute-homogeneous, and subadditive,

- $|x|_X \geq 0$ for all $x \in X$, with equality if and only if $x = 0$,
- $|ax|_X = |a|_{\mathbb{C}}|x|_X$ for all $a \in \mathbb{C}$ and $x \in X$,
- $|x + x'|_X \leq |x|_X + |x'|_X$ for all $x, x' \in X$.

The resulting metric is

$$d_X(x, x') = |x - x'|_X,$$

with the required metric properties in turn, i.e., it is positive, and symmetric and it satisfies the triangle inequality,

- $d_X(x, x') \geq 0$ for all $x, x' \in X$, with equality if and only if $x = x'$,
- $d_X(x, x') = d_X(x', x)$ for all $x, x' \in X$,
- $d_X(x, x') \leq d_X(x, x'') + d_X(x'', x')$ for all $x, x', x'' \in X$.

For any $x, x' \in X$ we have $|x|_X = |x - x' + x'|_X \leq |x - x'|_X + |x'|_X$, so that $|x|_X - |x'|_X \leq |x - x'|_X$, and symmetrically $|x'|_X - |x|_X \leq |x - x'|_X$. Thus $\big||x'|_X - |x|_X\big|_{\mathbb{R}} \leq |x - x'|_X$. This shows that the norm is uniformly continuous. The norm therefore extends uniquely to the completion of X. We will use this fact frequently.

For a more specific example, let X be the space of step functions from $[0, 1]$ to $\mathbb{R}$, meaning functions of the form

$$s = \sum_{i=1}^{n} c_i \chi_i,$$

with each χ_i the characteristic function of an interval I_i such that overall $[0, 1]$ is the disjoint union $\bigsqcup_{i=1}^{n} I_i$. The space X carries the norm

$$|s|_X = \max_i |c_i|_{\mathbb{R}}.$$

Define the integration function $\int_{[0,1]} : X \longrightarrow \mathbb{R}$,

$$\int_{[0,1]} s = \sum_i c_i \operatorname{length}(I_i) \quad \text{where} \quad s = \sum_i c_i \chi_i.$$

This function is linear, and it satisfies the condition

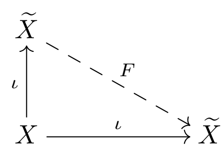
$$\left|\int_{[0,1]} s\right| = \left|\sum_i c_i \operatorname{length}(I_i)\right| \leq \sum_i |c_i| \operatorname{length}(I_i) \leq \max_i |c_i| = |s|_X,$$

which combines with its linearity to make it uniformly continuous. Therefore $\int_{[0,1]}$ extends to a function on the completion of X. This extension is the Riemann integral. If we start instead with the space Y of simple functions from $[0, 1]$ to $\mathbb{R}$, these being the finite linear combinations of characteristic functions of disjoint measurable subsets of $[0,1]$ rather than disjoint subintervals, then the resulting $\int_{[0,1]}$ extends to the Lebesgue integral on the completion of Y.

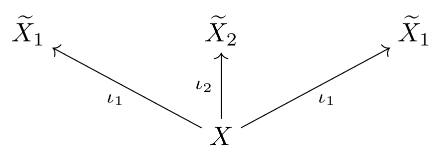
1. Uniqueness

Granting a completion, a standard argument shows that it is determined up to a unique uniformly continuous isomorphism with uniformly continuous inverse. Further we will see that this isomorphism is an isometry. That is, any two completions of a metric space X are related by a unique isometric isomorphism, which is to say that the completion is essentially unique.

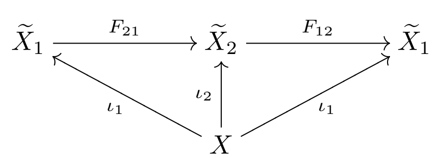
The standard argument is as follows. Let $\iota : X \longrightarrow \widetilde{X}$ be a completion of X. Let this completion take the role of $f : X \longrightarrow Z$ in the characterizing property. The property gives a unique F in the commutative diagram



Of course the identity map works, but the characterizing property says that *only* the identity map of $\widetilde{X}$ extends the isometry of X into $\widetilde{X}$ to a uniformly continuous map of $\widetilde{X}$ into itself. Next, if $\iota_j : X \longrightarrow \widetilde{X}_j$ for $j = 1, 2$ are completions of X then in the diagram



we give $\iota_1 : X \longrightarrow \widetilde{X}_1$ and $\iota_2 : X \longrightarrow \widetilde{X}_2$ respectively the roles of the completion and the auxiliary, and then exchange their roles, to get unique uniformly continuous functions $F_{21}, F_{1,2}$ in the commutative diagram



As noted above, this makes $F_{12} \circ F_{21}$ the identity map on $\widetilde{X}_1$. The same argument with 1 and 2 exchanged shows that $F_{21} \circ F_{12}$ is the identity map on $\widetilde{X}_2$. Thus $F_{21} : \widetilde{X}_1 \longrightarrow \widetilde{X}_2$ is a uniformly continuous isomorphism whose inverse is also uniformly continuous. Further, because ι_1 and ι_2 are isometries, also F_{21} is an isometry on $\iota_1(X)$.

Below we will show that $\iota_1(X)$ is dense in X_1, and that consequently F_{21} is an isometry on all of X_1. Granting this, we have shown that any two completion spaces $\widetilde{X}_1$ and $\widetilde{X}_2$ of X are isomorphic by a unique isometry, which is to say that the completion is determined up to unique isometric isomorphism.

2. Density

Assuming a completion $\iota : X \longrightarrow \widetilde{X}$, we show that X is dense in $\widetilde{X}$. Literally the statement is that $\iota(X)$ is dense in $\widetilde{X}$, i.e., any point in $\widetilde{X}$ is the limit of a sequence $\{\, \iota(x_i) \,\}$ in $\widetilde{X}$. Indeed, let $W \subset \widetilde{X}$ be the subspace of such limits, which includes $\iota(X)$ because each $x \in X$ gives the constant sequence $\{\, \iota(x) \,\}$. To show that W is complete, we must show that any Cauchy sequence $\{\, w_i \,\}$ in W converges in W. Since each w_i is the limit of a sequence in $\iota(X)$, there exists a sequence $\{\, x_i \,\}$ in X such that $d_W(\iota(x_i), w_i) < 2^{-i}$ for each i. To show that $\{\, w_i \,\}$ converges in W, we show that:

- Because $\{\, w_i \,\}$ is Cauchy in W, hence is Cauchy in $\widetilde{X}$, also $\{\, \iota(x_i) \,\}$ is Cauchy in $\widetilde{X}$, where it has a limit w. This limit lies in W by definition of W.
- Because $\{\, \iota(x_i) \,\}$ converges to w in W, so does $\{\, w_i \,\}$.

For the first bullet, given $\varepsilon > 0$ there exists i_o such that $2^{-i_o} < \varepsilon$ and $d_W(w_i, w_j) < \varepsilon$ for all $i, j \geq i_o$. Thus $\{\, \iota(x_i) \,\}$ is Cauchy because for all $i, j \geq i_o$,

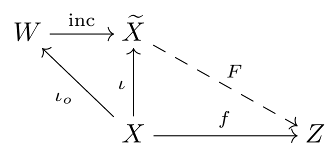
$$d_W(\iota(x_i), \iota(x_j)) \leq d_W(\iota(x_i), w_i) + d_W(w_i, w_j) + d_W(w_j, \iota(x_j)) < 3\varepsilon.$$

Similarly for the second bullet, given $\varepsilon > 0$ there exists i_o such that $2^{-i_o} < \varepsilon$ and $d_W(\iota(x_i), w) < \varepsilon$ for all $i \geq i_o$. Thus $\lim_i w_i = w$ because for all $i \geq i_o$,

$$d_W(w_i, w) \leq d_W(w_i, \iota(x_i)) + d_W(\iota(x_i), w) < 2\varepsilon.$$

This finishes the argument that W is complete.

Next we show that $\iota_o : X \longrightarrow W$, where ι_0 is ι with its codomain restricted from $\widetilde{X}$ to W, satisfies the completion mapping property, so that in fact W is all of $\widetilde{X}$. Indeed, letting *inc* denote the inclusion map, the diagram



shows that f induces the map $G = F \circ \mathrm{inc} : W \longrightarrow Z$, which inherits uniform continuity from F because the inclusion is an isometry. Furthermore, any induced map G out of W is determined by its values on $\iota_o(X)$, because $\iota_o(X)$ is dense in W by the construction of W and because G is continuous. More specifically, every point w of W takes the form $w = \lim_i \iota_o(x_i)$, and so

$$G(w) = G(\lim_i \iota_o(x_i)) = \lim_i G(\iota_o(x_i)).$$

Thus G is uniquely determined by f because $G \circ \iota_o = f$, and so $\iota_o : X \longrightarrow W$ satisfies the completion mapping property. The inclusion map from W to X is the isomorphism between the two completions, as established above, and so W is all of $\widetilde{X}$.

Now we can establish a fact that we used at the end of the previous section: Because X is dense in $\widetilde{X}$, if a map $f : X \longrightarrow Z$ from X to a complete metric space is an isometry, rather than just being uniformly continuous, then the resulting map $F : \widetilde{X} \longrightarrow Z$ is an isometry as well. This is because F is continuous, and the metrics d_Z and $d_{\widetilde{X}}$ are continuous (see below), and F is an isometry on $\iota(X)$ because $F \circ \iota = f$. Belaboring the matter by carefully avoiding limits in the noncomplete space X while using limits in the complete spaces $\widetilde{X}$ and Z, compute for any $\tilde{x}$ and $\tilde{x}'$ in $\widetilde{X}$,

$$\begin{aligned}
&d_Z(F(\tilde{x}), F(\tilde{x}'))\\
&\quad = d_Z(F(\lim_i \iota(x_i)), F(\lim_i \iota(x_i'))) &&\text{because } \iota(X) \text{ is dense in } \widetilde{X}\\
&\quad = d_Z(\lim_i F(\iota(x_i)), \lim_i F(\iota(x_i'))) &&\text{because } F \text{ is continuous}\\
&\quad = \lim_i d_Z(F(\iota(x_i)), F(\iota(x_i'))) &&\text{because } d_Z \text{ is continuous}\\
&\quad = \lim_i d_{\widetilde{X}}(\iota(x_i), \iota(x_i')) &&\text{because } F \text{ is an isometry on } \iota(X)\\
&\quad = d_{\widetilde{X}}(\lim_i \iota(x_i), \lim_i \iota(x_i')) &&\text{because } d_{\widetilde{X}} \text{ is continuous}\\
&\quad = d_{\widetilde{X}}(\tilde{x}, \tilde{x}').
\end{aligned}$$

To review that d_Z is continuous, let $\{ z_i \}$ and $\{ z_i' \}$ be convergent sequences in Z, having limits z and z'. Let $\varepsilon > 0$ be given. For any j and k we have

$$\begin{aligned}
d_Z(z, z') &\le d_Z(z, z_j) + d_Z(z_j, z_k') + d_Z(z_k', z')\\
d_Z(z_j, z_k') &\le d_Z(z_j, z) + d_Z(z, z') + d_Z(z', z_k'),
\end{aligned}$$

so that

$$|d_Z(z_j, z_k') - d_Z(z, z')| \le d_Z(z, z_j) + d_Z(z', z_k').$$

For large enough j and k we have $d_Z(z, z_j) < \varepsilon$ and $d_Z(z', z_k') < \varepsilon$, and so $|d_Z(z_j, z_k') - d_Z(z, z')| < 2\varepsilon$. This shows that $\lim_{j,k} d_Z(z_j, z_k') = d_Z(\lim_i z_i, \lim_i z_i')$, which is the continuity. Especially, $\lim_i d_Z(z_i, z_i') = d_Z(\lim_i z_i, \lim_i z_i')$, as in the computation above. This argument uses only general metric properties, so it applies to every metric space, and in particular to $\widetilde{X}$ along with Z.

3. Construction

3.1. **Pseudometric Space of Cauchy Sequences.** Let C be the space of Cauchy sequences in X. Let a typical element of C be $c = \{ x_j \}$. We show that the function

$$d : C \times C \longrightarrow \mathbb{R}_{\ge 0}, \qquad d_C(c, c') = \lim_j d_X(x_j, x_j')$$

is meaningful, i.e., the limit in the previous display exists. Given $\varepsilon > 0$ there exists $j_o(c)$ such that $d_X(x_j, x_k) < \varepsilon$ for all $j, k \ge j_o(c)$, and similarly for c'. Let $j_o = \max\{ j_o(c), j_o(c') \}$. For any indices $j, k \ge i_o$, the triangle inequality in X gives

$$|d_X(x_j, x_j') - d_X(x_k, x_k')| \le d_X(x_j, x_k) + d_X(x_j', x_k') < 2\varepsilon.$$

This shows that the real sequence $\{ d_X(x_j, x_j') \}$ is Cauchy, and hence it has a limit as claimed. Next, given sequences $c, c', c'' \in C$ and any index j, the triangle inequality in X gives $d_X(x_j, x_j') \le d_X(x_j, x_j'') + d_X(x_j'', x_j')$, and so passing to the limit gives the triangle inequality in C,

$$d_C(c, c') \le d_C(c, c'') + d_C(c'', c').$$

Also, d_C is obviously symmetric and nonnegative. However, it is a pseudometric rather than a metric because it is not strictly positive. The intuition is that $d_C(c, c') = 0$ if the Cauchy sequences c and c' are trying to converge to the same limit even if that limit does not exist. We will construct a true metric from d_C below.

3.2. **Completeness of C.** The proof that the pseudometric space C is complete is the main technical matter in all this. The argument is Cantor diagonalization. Let $\{ c_i \}$ be a Cauchy sequence in C. Thus each $c_i = \{ x_{i,j} \}$ is a Cauchy sequence in X. This is the famous, or infamous, *Cauchy sequence of Cauchy sequences* construct of analysis. Let $\varepsilon > 0$ be given. Since $\{ c_i \}$ is Cauchy in C,

$$\exists\, i(\varepsilon) \;:\; d_C(c_i, c_{i'}) < \varepsilon \quad \forall\, i, i' \geq i(\varepsilon),$$

and so, since $d_C(c_i, c_{i'}) = \lim_j d_X(x_{i',j}, x_{i,j})$ by definition,

$$(*) \qquad \forall\, i, i' \geq i(\varepsilon)\; \exists\, j(i,i') \;:\; d_X(x_{i',j}, x_{i,j}) < \varepsilon \quad \forall\, j \geq j(i,i').$$

Since each c_i is Cauchy in X,

$$(**) \qquad \forall\, i,\; \exists\, j(i) \;:\; d_X(x_{i,j}, x_{i,j'}) < \varepsilon \quad \forall\, j, j' \geq j(i).$$

We may take $j(i') \geq j(i), j(i,i')$ for all $i' > i \geq i(\varepsilon)$. Define a diagonal-like sequence,

$$\ell = \{ x_{i,j(i)} \}.$$

Visually, ℓ is something like this:

$$\begin{array}{cccccccccc}
x_{1,1} & \boxed{x_{1,2}} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} & \cdots \\
x_{2,1} & x_{2,2} & x_{2,3} & \boxed{x_{2,4}} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} & \cdots \\
x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & \boxed{x_{3,5}} & x_{3,6} & x_{3,7} & x_{3,8} & x_{3,9} & \cdots \\
x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} & x_{4,6} & x_{4,7} & \boxed{x_{4,8}} & x_{4,9} & \cdots \\
\cdots & & & & & & & & &
\end{array}$$

To see that ℓ is Cauchy, note that for any $i' > i \geq i(\varepsilon)$, the distance between its i'th and ith terms satisfies

$$d_X(\ell_{i'}, \ell_i) = d_X(x_{i',j(i')}, x_{i,j(i)}) \leq d_X(x_{i',j(i')}, x_{i,j(i')}) + d_X(x_{i,j(i')}, x_{i,j(i)}).$$

The first term is less than ε by $(*)$ because $j(i') \geq j(i,i')$, and the second term is less than ε by $(**)$ because $j(i), j(i') \geq j(i)$. So indeed ℓ is Cauchy. To see that $\ell = \lim_i c_i$, take any $i \geq i(\varepsilon)$ and note that for any $i' \geq i, j(i)$, the distance between the i'th terms of ℓ and c_i satisfies

$$d_X(\ell_{i'}, x_{i,i'}) = d_X(x_{i',j(i')}, x_{i,i'}) \leq d_X(x_{i',j(i')}, x_{i,j(i')}) + d_X(x_{i,j(i')}, x_{i,i'}).$$

Again the first term is less than ε by $(*)$ because $j(i') \geq j(i,i')$, and the second term is less than ε by $(**)$ because $j(i'), i' \geq j(i)$. So indeed $\ell = \lim_i c_i$.

3.3. **Mapping Property of C.** Now we show that the pseudometric space C satisfies the mapping property of the completion other than not being a true metric space. The isometry from X to C takes each element to the corresponding constant sequence. Given a uniformly continuous function $f : X \longrightarrow Z$ where Z is complete, the only possible compatible induced uniformly continuous function is

$$F : C \longrightarrow Z, \qquad F(c) = \lim_j f(x_j) \quad \text{where } c = \{ x_j \}.$$

The limit exists because the sequence $\{ f(x_j) \}$ is Cauchy in Z in consequence of the sequence $\{ x_j \}$ being Cauchy in X and f being uniformly continuous, and because Z is complete. With F now well defined, also it is uniformly continuous, as follows. Let $\varepsilon > 0$ be given, and let $\delta = \delta(f, \varepsilon/2)$. For Cauchy sequences $c = \{ x_j \}$ and

$c' = \{ x'_j \}$ in C with $d_C(c, c') < \delta$, we have $d_X(x_j, x'_j) < \delta$ for all large enough j, and so $d_Z(f(x_j), f(x'_j)) < \varepsilon/2$ for all large enough j. Thus

$$d_Z(F(c), F(c')) = d_Z(\lim_j f(x_j), \lim_i f(x'_j)) = \lim_j d_Z(f(x_j), f(x'_j)) \leq \varepsilon/2 < \varepsilon.$$

This shows that F is uniformly continuous as desired.

In particular, if f is a norm function $|\cdot|_X : X \longrightarrow \mathbb{R}$, then

- $F(c) = \lim_j |x_j|_X \geq 0$, with equality if and only if $\lim_j |x_j|_X = 0$,
- $F(ac) = \lim_j |ax_j|_X = \lim_j |a|_{\mathbb{C}}|x_j|_X = |a|_{\mathbb{C}} \lim_j |x_j|_X$, so F is absolute-homogeneous,
- $F(c+c') = \lim_j |x_j+x'_j|_X \leq \lim_j(|x_j|_X+|x'_j|_X) = \lim_j |x_j|_X+\lim_j |x'_j|_X = F(c) + F(c')$, so F is subadditive.

This F is a pseudonorm, failing only the strict positivity condition.

3.4. **True Metric Space $\widetilde{X} = C/\sim$, Its Mapping Property.** Finally, to repair the problem that C is only a pseudometric space, construct its quotient space

$$\widetilde{X} = C/\sim,$$

where the equivalence relation is

$$c' \sim c \iff d_C(c, c') = 0,$$

and the elements of the quotient space are equivalence classes,

$$[c] = \{ c' \in C : c' \sim c \}.$$

Especially, if X is a linear space then the class $[\{0, 0, 0, \dots\}]$ consists of all Cauchy sequences that converge to 0. We show that d_C is defined on pairs of classes. Suppose that $c'_1 \sim c_1$ in C and take any c_2 in C. Thus

$$d_C(c_1, c_2) \leq d_C(c_1, c'_1) + d_C(c'_1, c_2) = d_C(c'_1, c_2),$$

and similarly $d_C(c'_1, c_2) \leq d_C(c_1, c_2)$, so in fact $d_C(c'_1, c_2) = d_C(c_1, c_2)$. Symmetrically, $d_C(c_1, c'_2) = d_C(c_1, c_2)$ if $c'_2 \sim c_2$. Thus $d_{\widetilde{X}}([c_1], [c_2])$ is well defined as the common value of $d_C(c'_1, c'_2)$ over all $c'_1 \in [c_1]$ and $c'_2 \in [c_2]$. And if $d_{\widetilde{X}}([c_1], [c_2]) = 0$ then $[c_1] = [c_2]$, so $d_{\widetilde{X}}$ is a true metric. The projection map $\pi : C \longrightarrow \widetilde{X}$ taking each $c \in C$ to $[c]$ is an isometry.

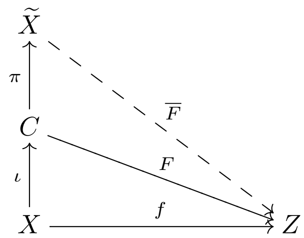
To see that the mapping property of the completion holds for $\widetilde{X}$, we check that any induced map out of C is constant on equivalence classes and therefore factors uniquely through $\widetilde{X}$. Let $\varepsilon > 0$ be given, and let $\delta = \delta(f, \varepsilon)$. For Cauchy sequences $c = \{ x_j \}$ and $c' = \{ x'_j \}$, the condition $c \sim c'$ is $d_C(c, c') = 0$, or $\lim_j d_X(x_j, x'_j) = 0$, giving

$$d_X(x_j, x'_j) < \delta \text{ for all large enough } j,$$

so that

$$d_Z(f(x_j), f(x'_j)) < \varepsilon \text{ for all large enough } j.$$

Thus $\lim_j d_Z(f(x_j), f(x'_j)) = 0$. Consequently, $d_Z(\lim_j f(x_j), \lim_j f(x'_j)) = 0$, so that $\lim_j f(x_j) = \lim_j f(x'_j)$, meaning precisely that $F(c) = F(c')$. Thus F is well defined on classes as claimed, and the definition $\overline{F}([c]) = F(c)$ is sensible. So $\overline{F} \circ \pi = F$, and we have the desired diagram



The map $\overline{F} : \widetilde{X} \longrightarrow Z$ is uniformly continuous in consequence of $F : C \longrightarrow Z$ being uniformly continuous, because the projection $\pi : C \longrightarrow \widetilde{X}$ is an isometry.

As a last comment, we note that for many purposes it suffices to have the completion definition posit that f and F are subisometries, meaning that $d_Z(f(x), f(x')) \leq d_X(x, x')$ for all $x, x' \in X$ and similarly for F, rather than f and F being uniformly continuous functions. With this variant characterizing property, some of the arguments of this writeup become a little easier. In particular, the argument that if the completion exists then it is determined up to unique isometry doesn't depend on the density of the space in its completion.