

## THE FOURIER TRANSFORM AND THE MELLIN TRANSFORM

For suitable functions

$$f : \mathbf{R} \longrightarrow \mathbf{C}$$

the *Fourier transform of  $f$*  is the integral

$$g : \mathbf{R} \longrightarrow \mathbf{C}, \quad g(y) = \int_{\mathbf{R}} f(x)e^{2\pi ixy} dx,$$

and for suitable functions

$$g : \mathbf{R} \longrightarrow \mathbf{C}$$

the *inverse Fourier transform of  $g$*  is the integral

$$f : \mathbf{R} \longrightarrow \mathbf{C}, \quad f(x) = \int_{\mathbf{R}} g(y)e^{-2\pi iyx} dy.$$

The *Fourier inversion formula* says that if the functions  $f$  and  $g$  are well enough behaved then  $g$  is the Fourier transform of  $f$  if and only if  $f$  is the inverse Fourier transform of  $g$ .

The exponential map is a topological isomorphism

$$\exp : (\mathbf{R}, +) \longrightarrow (\mathbf{R}^+, \cdot)$$

The Mellin transform, inverse Mellin transform, and Mellin inversion formula are essentially the Fourier ideas passed through the isomorphism.

Specifically, given a suitable function on the positive real axis,

$$f : \mathbf{R}^+ \longrightarrow \mathbf{C},$$

we can make a corresponding function on the real line,

$$\tilde{f} : \mathbf{R} \longrightarrow \mathbf{C}, \quad \tilde{f} = f \circ \exp.$$

The Fourier transform of  $\tilde{f}$  is  $\tilde{g} : \mathbf{R} \longrightarrow \mathbf{C}$  where

$$\begin{aligned} \tilde{g}(y) &= \int_{\mathbf{R}} \tilde{f}(x)e^{2\pi ixy} dx \\ &= \int_{\mathbf{R}} f(e^x)(e^x)^{2\pi iy} \frac{d(e^x)}{e^x} \\ &= \int_{\mathbf{R}^+} f(t)t^{2\pi iy} \frac{dt}{t} \\ &= \int_{\mathbf{R}^+} f(t)t^s \frac{dt}{t} \quad \text{where } s = 2\pi iy. \end{aligned}$$

If we assume that our function  $f(t)$  decreases at least as a polynomial in  $t$  as  $t \rightarrow 0^+$  and that  $f$  decreases quickly as  $t \rightarrow \infty$  then in fact the integral converges for some complex right half plane of  $s$ -values  $\{\operatorname{Re}(s) > \sigma_0\}$  where  $\sigma_0 < 0$ . Thus we are led to define the *Mellin transform of  $f$* ,

$$g : \{\operatorname{Re}(s) > \sigma_0\} \longrightarrow \mathbf{C}, \quad g(s) = \int_{\mathbf{R}^+} f(t)t^s \frac{dt}{t}.$$

The next question is how to recover  $f$  from  $g$ . Since  $g$  is simply the Fourier transform of  $f$  up to a coordinate change,  $f$  must be essentially the inverse Fourier transform of  $g$ . More specifically, the fact that  $\tilde{f}$  is exactly the inverse Fourier transform of  $\tilde{g}$ ,

$$\tilde{f}(x) = \int_{\mathbf{R}} \tilde{g}(y) e^{-2\pi i y x} dy,$$

rewrites as

$$f(e^x) = \frac{1}{2\pi i} \int_{s=2\pi i y} g(s) (e^x)^{-s} ds.$$

That is,

$$f(t) = \frac{1}{2\pi i} \int_{s=2\pi i y} g(s) t^{-s} ds.$$

Contour integration shows that the vertical line of integration can be shifted horizontally within the right half plane of convergence with no effect on the integral. Thus the definition of the *inverse Mellin transform* of  $g$  is inevitably

$$f : \mathbf{R}^+ \longrightarrow \mathbf{C}, \quad f(t) = \frac{1}{2\pi i} \int_{s=\sigma+2\pi i y} g(s) t^{-s} ds \quad \text{for any suitable } \sigma.$$

Here it is understood that the integral proceed *up* the vertical line.

Naturally, the *Mellin inversion formula* says that if the functions  $f$  and  $g$  are well enough behaved then  $g$  is the Mellin transform of  $f$  if and only if  $f$  is the inverse Mellin transform of  $g$ . For practice with Mellin inversion, it is an exercise to evaluate the integral

$$\int_{s=\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) x^{-s} ds, \quad \sigma > 0.$$

Often what this writeup calls the Fourier transform is called the inverse Fourier transform and conversely. It is a small exercise to show that which convention is adopted has no effect on the resulting definition of the Mellin transform and the inverse Mellin transform.