

## LOCAL FACTORS OF ZETA FUNCTIONS

We have seen that the functional equation of the Euler–Riemann zeta function is

$$\xi(1-s) = \xi(s), \quad s \in \mathbf{C},$$

where the *initial* definition of  $\xi$  is

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \operatorname{Re}(s) > 1.$$

The Euler product expression of  $\zeta(s)$  is

$$\zeta(s) = \prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1}.$$

And we have seen similar formulas in the context of Dirichlet  $L$ -functions.

This writeup sketches the calculation that the *Archimedean factors* such as  $\pi^{-s/2} \Gamma(s/2)$  and the *non-Archimedean factors* such as  $(1 - p^{-s})^{-1}$  have a uniform description.

### 1. THE LOCAL ZETA INTEGRAL

Let  $\mathbf{k}$  be any of the following fields:

$$\begin{cases} \mathbf{k} = \mathbf{Q}_p \text{ for some prime } p, \\ \mathbf{k} = \mathbf{R}, \\ \mathbf{k} = \mathbf{C}. \end{cases}$$

The **Schwartz space** of  $\mathbf{k}$  is a collection of nice functions

$$\varphi : \mathbf{k} \longrightarrow \mathbf{C}.$$

Specifically, the Schwartz space consists of the smooth functions on  $\mathbf{k}_v$  all of whose derivatives are rapidly decreasing. If  $\mathbf{k} = \mathbf{Q}_p$  then such functions are compactly supported.

Take a Schwartz function

$$\varphi : \mathbf{k} \longrightarrow \mathbf{C},$$

and take a continuous multiplicative character

$$\chi : \mathbf{k}^\times \longrightarrow \mathbf{C}^\times.$$

Let  $s$  be a complex parameter. Then the **local zeta integral** is

$$Z(s) = Z(\varphi, \chi, s) = \int_{\mathbf{k}^\times} |\alpha|^s \chi(\alpha) \varphi(\alpha) d\alpha.$$

The integral converges for all  $s$  in some open right half plane.

## 2. NONARCHIMEDEAN PLACES

Let  $\mathbf{k} = \mathbf{Q}_p$  for some prime  $p$ . The ring of integers of  $\mathbf{k}$  is  $\mathbf{Z}_p$ . Consider a character

$$\chi : \mathbf{Q}_p^\times \longrightarrow \mathbf{C}^\times, \quad \chi(\alpha) = |\alpha|^{s_0}$$

(in the context of a primitive Dirichlet character, these local characters arise for the primes  $p$  away from the Dirichlet character's conductor), and consider a Schwartz function

$$\varphi : \mathbf{Q}_p \longrightarrow \mathbf{C}, \quad \varphi = 1_{\mathbf{Z}_p}.$$

This *locally constant* function is smooth, due to the non-Archimedean nature of  $\mathbf{Q}_p$ . The local zeta integral is

$$Z(s) = \int_{\mathbf{Q}_p^\times} |\alpha|^s \chi(\alpha) \varphi(\alpha) d\alpha$$

Normalize the multiplicative Haar measure so that  $\mu(\mathbf{Z}_p^\times) = 1$ . Evaluating the local integral is straightforward,

$$Z(s) = \int_{\mathbf{Q}_p^\times / \mathbf{Z}_p^\times} \int_{\mathbf{Z}_p^\times} |\alpha\eta|^{s+s_0} \varphi(\alpha\eta) d\eta d\alpha = \sum_{\ell=0}^{\infty} |p^\ell|^{(s+s_0)} = (1 - \chi(p)p^{-s})^{-1}.$$

Thus the Euler factor of  $\zeta(s)$  or of  $L(s, \chi)$  at  $p$  is the  $p$ -adic zeta integral.

## 3. REAL ARCHIMEDEAN PLACES

Let  $\mathbf{k} = \mathbf{R}$ . Consider the trivial character and the Gaussian Schwartz function

$$\begin{aligned} \chi : \mathbf{R}^\times &\longrightarrow \mathbf{C}^\times, & \chi(\alpha) &= 1, \\ \varphi : \mathbf{R} &\longrightarrow \mathbf{C}, & \varphi(\alpha) &= e^{-\pi\alpha^2}. \end{aligned}$$

The local integral is

$$Z(s) = \int_{\mathbf{R}^\times} |\alpha|^s \chi(\alpha) \varphi(\alpha) d\alpha = 2 \int_0^\infty t^s e^{-\pi t^2} \frac{dt}{t} = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

On the other hand, consider the sign character and the simplest odd Schwartz function that incorporates the Gaussian,

$$\begin{aligned} \chi : \mathbf{R}^\times &\longrightarrow \mathbf{C}^\times, & \chi(\alpha) &= \text{sgn}(\alpha), \\ \varphi : \mathbf{R} &\longrightarrow \mathbf{C}, & \varphi(\alpha) &= \alpha e^{-\pi\alpha^2}. \end{aligned}$$

Then the local integral is

$$Z(s) = \int_{\mathbf{R}^\times} |\alpha|^s \chi(\alpha) \varphi(\alpha) d\alpha = 2 \int_0^\infty t^{s+1} e^{-\pi t^2} \frac{dt}{t} = \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

Thus the remaining factor in the functional equation of  $\zeta(s)$  or of  $L(s, \chi)$  is the Archimedean zeta integral.

## 4. COMPLEX ARCHIMEDEAN PLACES

Let  $\mathbf{k} = \mathbf{C}$ . The unitary characters are

$$\chi : \mathbf{C}^\times \longrightarrow \mathbf{T}, \quad \chi(\alpha) = \left( \frac{\alpha}{|\alpha|} \right)^m \text{ where } m \in \mathbf{Z}.$$

The Haar measure is

$$d\alpha = \frac{d^+\alpha}{|\alpha|^2} = \frac{r dr d\theta}{r^2}.$$

Note that the change of variable  $\rho = r^2$  gives  $2r dr/r^2 = d\rho/\rho$ .

If  $m = 0$  then let  $\varphi(\alpha) = e^{-\pi|\alpha|^2}$ . The local integral is

$$Z(s) = \int_{\mathbf{C}^\times} |\alpha|_{\mathbf{C}}^s \chi(\alpha) \varphi(\alpha) d\alpha = 2\pi \int_0^\infty r^{2s} e^{-\pi r^2} \frac{r dr}{r^2} = \pi \cdot \pi^{-s} \Gamma(s).$$

If  $m > 0$  then let  $\varphi(\alpha) = \bar{\alpha}^m e^{-\pi|\alpha|^2}$ . The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbf{C}^\times} |\alpha|_{\mathbf{C}}^s \chi(\alpha) \varphi(\alpha) d\alpha = 2\pi \int_0^\infty r^{2s+m} e^{-\pi r^2} \frac{r dr}{r^2} \\ &= \pi \cdot \pi^{-s - \frac{m}{2}} \Gamma\left(s + \frac{m}{2}\right). \end{aligned}$$

If  $m < 0$  then let  $\varphi(\alpha) = \alpha^{-m} e^{-\pi|\alpha|^2}$ . The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbf{C}^\times} |\alpha|_{\mathbf{C}}^s \chi(\alpha) \varphi(\alpha) d\alpha = 2\pi \int_0^\infty r^{2s-m} e^{-\pi r^2} \frac{r dr}{r^2} \\ &= \pi \cdot \pi^{-s + \frac{m}{2}} \Gamma\left(s - \frac{m}{2}\right). \end{aligned}$$

Thus in all cases,

$$Z(s) = \pi \cdot \pi^{-s - \frac{|m|}{2}} \Gamma\left(s + \frac{|m|}{2}\right).$$

This zeta integral is not something that we recognize, but it occurs naturally in the context of number fields larger than  $\mathbf{Q}$ .

The point of this writeup is *not* that the zeta integral artificially repackages the various Euler factors in a uniform way, but rather that the specific choices of characters and Schwartz functions that reproduced the familiar Euler factors are in fact mere placeholders.