NONVANISHING OF DIRICHLET L-FUNCTIONS AT $s = 1$

In the proof of Dirichlet's theorem on arithmetic progressions, after the various sums and products are unwound, and after what amounts to a simple piece of Fourier analysis, the crucial fact is that for any nontrivial Dirichlet character χ,

$$L(\chi, s) \neq 0 \quad \text{at } s = 1.$$

The fact can be proved in various ways. For example, our handout on Dirichlet's theorem made use of cyclotomic arithmetic. Here we give, with some motivation, a more direct elementary argument, which admittedly is a bit *ad hoc*.

Contents

1. The argument when χ^2 is nontrivial

For any $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$,

$$\begin{aligned} L(\chi, s) &= \exp \log L(\chi, s) = \exp \log \prod_p (1 - \chi(p)p^{-s})^{-1} \\ &= \exp \sum_{p \in \mathcal{P}} \log(1 - \chi(p)p^{-s})^{-1} = \exp \sum_{p \in \mathcal{P}} \sum_{n \in \mathbb{Z}^+} \frac{\chi(p)^n}{np^{ns}}. \end{aligned}$$

Because in general $|\exp(z)| = \exp(\text{Re}(z))$, it follows that for *real* $s > 1$,

$$|L(\chi, s)| = \exp \sum_{p,n} \frac{\cos(n\theta_p)}{np^{ns}} \quad \text{where } \chi(p) = e^{i\theta_p}.$$

The cosines in the sum could well be positive or negative. However, modifying the calculation makes the summands nonnegative,

$$\begin{aligned} \zeta(s)\, L(\chi, s) &= \exp \log\Big(\prod_p (1 - p^{-s})^{-1}(1 - \chi(p)p^{-s})^{-1}\Big) \\ &= \exp \sum_{p \in \mathcal{P}} \log(1 - p^{-s})^{-1} + \log(1 - \chi(p)p^{-s})^{-1} \\ &= \exp \sum_{p \in \mathcal{P}} \sum_{n \in \mathbb{Z}^+} \frac{1 + \chi(p)^n}{np^{ns}}, \quad s > 1, \end{aligned}$$

so that

$$|\zeta(s)\, L(\chi, s)| = \exp \sum_{p,n} \frac{1 + \cos(n\theta_p)}{np^{ns}}, \quad s > 1.$$

Now the summands are nonnegative, and thus

$$|\zeta(s)\,L(\chi,s)| \geq 1, \quad s>1.$$

This doesn't give $L(\chi,1)\neq 0$, though, because ζ has a simple pole at $s=1$, and so the previous display shows only that $L(\chi,s)$ either is nonzero at $s=1$ or has a simple zero at $s=1$. Because a zero would force $\zeta(s)L(\chi,s)^2$ to vanish at $s=1$, the next step is to study

$$|\zeta(s)\,L(\chi,s)^2| = \exp\sum_{p,n}\frac{1+2\cos(n\theta_p)}{np^{ns}}\,, \quad s>1,$$

but now we are back to a scenario where the terms of the sum need not be positive. To address this, the expression $1+2\cos(n\theta_p)$ can be augmented to a square by adding $\cos^2(n\theta_p)$. Thus, consider

$$L(\chi^2,s) = \exp\sum_{p,n}\frac{(\chi(p)^n)^2}{np^{ns}}\,,$$

so that, because $\chi(p)^n = \cos(n\theta_p)+i\sin(n\theta_p)$, and hence $(\chi(p)^n)^2$ has real part $\cos^2(n\theta_p)-\sin^2(n\theta_p) = 2\cos^2(n\theta_p)-1$,

$$|L(\chi^2,s)| = \exp\sum_{p,n}\frac{2\cos^2(n\theta_p)-1}{np^{ns}}\,, \quad s>1.$$

Thus, more generally,

$$|\zeta(s)^a L(\chi,s)^b L(\chi^2,s)^c| = \exp\sum_{p,n}\frac{a-c+b\cos(n\theta_p)+2c\cos^2(n\theta_p)}{np^{ns}}\,, \quad s>1.$$

Specialize to $(a,b,c)=(3,4,1)$ to get

$$\begin{aligned}|\zeta(s)^3 L(\chi,s)^4 L(\chi^2,s)| &= \exp\sum_{p,n}\frac{2+4\cos(n\theta_p)+2\cos^2(n\theta_p)}{np^{ns}}\\ &= \exp\sum_{p,n}\frac{2(1+\cos(n\theta_p))^2}{np^{ns}} \geq 1, \quad s>1,\end{aligned}$$

so that

$$\zeta(s)^3 L(\chi,s)^4 L(\chi^2,s) \quad \text{does not go to 0 as } s\to 1^+.$$

But $\zeta(s)^3$ has a pole of order 3 at $s=1$, and assuming that χ^2 is not the trivial character, $L(\chi^2,s)$ does not have a pole at $s=1$. So the previous display shows that $L(\chi,s)$ can't have a zero at $s=1$ if χ^2 is nontrivial.

2. The argument when χ^2 is trivial

The case where χ^2 is trivial needs to be handled separately. Here we have

$$\zeta(s)\,L(\chi,s) = \exp\sum_{p,n}\frac{1+\chi(p)^n}{np^{ns}}\,, \quad \mathrm{Re}(s)>1.$$

The sum in the previous display is a Dirichlet series $D(s)$ with nonnegative coefficients,

$$D(s) = \sum_{m\in\mathbb{Z}^+}\frac{a_m}{m^s}, \quad a_m = \begin{cases}(1+\chi(p)^n)/n & \text{if } m=p^n,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose that $L(\chi, 1) = 0$. Then consequently:

- The function $\zeta(s)\, L(\chi, s)$ is analytic on $\{\mathrm{Re}(s) > 0\}$.
- The Dirichlet series $D(s)$ converges on $\mathrm{Re}(s) > 1$ to a function $g(s)$ such that $\exp g(s) = \zeta(s)\, L(\chi, s)$. Landau's lemma, below, says that consequently $\exp D(s) = \zeta(s)\, L(\chi, s)$ for $s \in (0, 1)$.
- However, when n is even, $\chi(p)^n = 1$, and so for real $s > 1/2$,
$$D(s) \geq \sum_{p,n} \frac{2}{2np^{2ns}} = \sum_{p,n} \frac{1}{np^{2ns}} = \log \zeta(2s).$$
Thus $D(s) \to \infty$ as $s \to 1/2^+$.

The third bullet contradicts the second, so the supposition $L(\chi, 1) = 0$ is untenable.

3. Landau's lemma

Proposition 3.1 (Weak version of Landau's lemma)**.** *Suppose that a Dirichlet series with nonnegative coefficients,*
$$D(s) = \sum_{n \geq 1} a_n n^{-s}, \quad a_n \geq 0 \text{ for all } n,$$
converges to an analytic function $f(s)$ on the open right half plane $\{\mathrm{Re}(s) > \sigma_o\}$. Suppose that for some $\varepsilon > 0$, the function $f(s)$ extends analytically to the larger open right half plane $\{\mathrm{Re}(s) > \sigma_o - \varepsilon\}$. Then the Dirichlet series $D(s)$ converges to $f(s)$ on the x-axis portion of the larger right half plane, i.e., $D(\sigma) = f(\sigma)$ for all $\sigma \in (\sigma_o - \varepsilon, \sigma_o)$.

Proof. By way of quick review, recall the basic definition
$$a^z = e^{z \log a}, \quad a \in \mathbb{R}^+,\ z \in \mathbb{C},$$
so that the derivatives of a^z are
$$(a^z)^{(k)} = (\log a)^k a^z, \quad a \in \mathbb{R}^+,\ z \in \mathbb{C},\ k \in \mathbb{Z}_{\geq 0}.$$
Thus the power series expansion of a^z about $z = 0$ is
$$a^z = \sum_{k \geq 0} \frac{(\log a)^k}{k!} z^k, \quad a \in \mathbb{R}^+,\ z \in \mathbb{C},$$
Note that this is a small variant of the familiar series of e^z. We will refer back to this expansion later in the argument.

Returning to Landau's lemma, we may translate the problem and take $\sigma_o = 0$. The translation leaves the Dirichlet series coefficients nonnegative. The function $f(s)$ is analytic on $B(1, 1+\varepsilon)$. Thus for any $\sigma \in (-\varepsilon, 0)$ the power series representation of $f(s)$ about $s = 1$ converges at σ to $f(\sigma)$,
$$f(\sigma) = \sum_{k \geq 0} \frac{f^{(k)}(1)}{k!} (\sigma - 1)^k = \sum_{k \geq 0} \frac{(-1)^k f^{(k)}(1)}{k!} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0.$$
Because the Dirichlet series $D(s) = \sum_{n \geq 1} a_n n^{-s}$ converges to $f(s)$ about $s = 1$, compute the summand-numerator $(-1)^k f^{(k)}(1)$ at the end of the previous display by differentiating $D(s)$ termwise,
$$(-1)^k f^{(k)}(1) = (-1)^k \sum_{n \geq 1} a_n (-\log n)^k n^{-s} \Big|_{s=1} = \sum_{n \geq 1} \frac{a_n (\log n)^k}{n}.$$

Thus the penultimate display is now

$$f(\sigma) = \sum_{k\geq 0}\sum_{n\geq 1} \frac{a_n(\log n)^k}{k!n}(1-\sigma)^k, \quad -\varepsilon < \sigma < 0.$$

All of the terms are nonnegative, so we may rearrange the sum,

$$f(\sigma) = \sum_{n\geq 1} \frac{a_n}{n} \sum_{k\geq 0} \frac{(\log n)^k}{k!}(1-\sigma)^k, \quad -\varepsilon < \sigma < 0.$$

As explained at the beginning of the proof, the inner sum is the power series expansion of n^s about 0 at $s = 1 - \sigma$. Thus

$$f(\sigma) = \sum_{n\geq 1} \frac{a_n}{n}\, n^{1-\sigma} = \sum_{n\geq 1} a_n n^{-\sigma} = D(\sigma), \quad -\varepsilon < \sigma < 0.$$

This is the desired result. □