NONVANISHING OF DIRICHLET L-FUNCTIONS AT $s = 1$

In the proof of Dirichlet's theorem on arithmetic progressions, after the various sums and products are unwound, and after what amounts to a simple piece of Fourier analysis, the crucial fact is that for any nontrivial Dirichlet character χ,

$$L(s,\chi) \neq 0 \quad \text{at } s = 1.$$

The fact can be proved in various ways. For example, our handout on Dirichlet's theorem made use of cyclotomic arithmetic. Here we give, with some motivation, a more direct elementary argument, which admittedly is a bit *ad hoc*.

1. The Standard Trick When χ^2 is Nontrivial

For any $s \in \mathbf{C}$ such that $\mathrm{Re}(s) > 1$,

$$\begin{aligned} L(s,\chi) &= \exp\log L(s,\chi) = \exp\log \prod_p (1-\chi(p)p^{-s})^{-1} \\ &= \exp \sum_{p\in\mathcal{P}} \log(1-\chi(p)p^{-s})^{-1} = \exp \sum_{p\in\mathcal{P}} \sum_{n\in\mathbf{Z}^+} \frac{\chi(p)^n}{np^{ns}}. \end{aligned}$$

Since in general $|\exp(z)| = \exp(\mathrm{Re}(z))$, it follows that for *real* $s > 1$,

$$|L(s,\chi)| = \exp \sum_{p,n} \frac{\cos(n\theta_p)}{np^{ns}} \quad \text{where } \chi(p) = e^{i\theta_p}.$$

The cosines in the sum could well be positive or negative. However, modifying the calculation makes the summands nonnegative,

$$\begin{aligned} \zeta(s)\,L(s,\chi) &= \exp\log(\prod_p (1-p^{-s})^{-1}(1-\chi(p)p^{-s})^{-1}) \\ &= \exp \sum_{p\in\mathcal{P}} \log(1-p^{-s})^{-1} + \log(1-\chi(p)p^{-s})^{-1} \\ &= \exp \sum_{p\in\mathcal{P}} \sum_{n\in\mathbf{Z}^+} \frac{1+\chi(p)^n}{np^{ns}}, \quad s > 1, \end{aligned}$$

so that

$$|\zeta(s)\,L(s,\chi)| = \exp \sum_{p,n} \frac{1+\cos(n\theta_p)}{np^{ns}}, \quad s > 1.$$

Now the summands are nonnegative, and thus

$$|\zeta(s)\,L(s,\chi)| \geq 1, \quad s > 1.$$

However, since ζ has a simple pole at $s = 1$, the previous display shows only that $L(s,\chi)$ either is nonzero at $s = 1$ or has a simple zero at $s = 1$. Since a zero would force $\zeta(s)L(s,\chi)^2$ to vanish at $s = 1$, the next step is to study

$$|\zeta(s)\,L(s,\chi)^2| = \exp \sum_{p,n} \frac{1+2\cos(n\theta_p)}{np^{ns}}, \quad s > 1,$$

but now we are back to a scenario where the terms of the sum need not be positive. However, the expression $1 + 2\cos(n\theta_p)$ clearly wants to have the square completed. So now consider

$$L(s, \chi^2) = \exp \sum_{p,n} \frac{(\chi(p)^n)^2}{np^{ns}},$$

so that, since $\chi(p)^n = \cos(n\theta_p) + i\sin(n\theta_p)$,

$$\begin{aligned} |L(s, \chi^2)| &= \exp \sum_{p,n} \frac{\cos^2(n\theta_p) - \sin^2(n\theta_p)}{np^{ns}} \\ &= \exp \sum_{p,n} \frac{2\cos^2(n\theta_p) - 1}{np^{ns}}, \quad s > 1, \end{aligned}$$

Thus, more generally,

$$|\zeta(s)^a L(s, \chi)^b L(s, \chi^2)^c| = \exp \sum_{p,n} \frac{a - c + b\cos(n\theta_p) + 2c\cos^2(n\theta_p)}{np^{ns}}, \quad s > 1.$$

Especially, taking $a = 3$, $b = 4$, $c = 1$ gives

$$\begin{aligned} |\zeta(s)^3 L(s, \chi)^4 L(s, \chi^2)| &= \exp \sum_{p,n} \frac{2 + 4\cos(n\theta_p) + 2\cos^2(n\theta_p)}{np^{ns}} \\ &= \exp \sum_{p,n} \frac{2(1 + \cos(n\theta_p))^2}{np^{ns}} \geq 1, \quad s > 1, \end{aligned}$$

so that

$$\zeta(s)^3 L(s, \chi)^4 L(s, \chi^2) \quad \text{does not go to 0 as } s \to 1^+.$$

But $\zeta(s)^3$ has a pole of order 3 at $s = 1$, and *assuming that χ^2 is not the trivial character*, $L(s, \chi^2)$ does not have a pole at $s = 1$. So the previous display shows that $L(s, \chi)$ can not have a zero at $s = 1$ if χ^2 is nontrivial..

2. The Trick When χ^2 is Trivial

The case where χ^2 is trivial needs to be handled separately. Here we have

$$\zeta(s)\, L(s, \chi) = \exp \sum_{p,n} \frac{1 + \chi(p)^n}{np^{ns}}, \quad \mathrm{Re}(s) > 1.$$

Note that the Dirichlet series in the previous display has nonnegative coefficients. Specifically, it is

$$D(s) = \sum_{p,n} \frac{1 + \chi(p)^n}{np^{ns}} = \sum_{m \in \mathbf{Z}^+} \frac{a_m}{m^s}, \quad a_m = \begin{cases} (1 + \chi(p)^n)/n & \text{if } m = p^n, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $L(1, \chi) = 0$. Then consequently:

- The function $\zeta(s)\, L(s, \chi)$ is analytic on $\{\mathrm{Re}(s) > 0\}$.
- The Dirichlet series $D(s)$ converges on $\mathrm{Re}(s) > 1$ to a function $g(s)$ such that $\exp g(s) = \zeta(s)\, L(s, \chi)$. Landau's Lemma (see below) says that consequently $\exp D(s) = \zeta(s)\, L(s, \chi)$ for $s \in (0, 1)$.

- However, when n is even, $\chi(p)^n = 1$, and so for real $s > 1/2$,
$$D(s) \geq \sum_{p,n} \frac{2}{2np^{2ns}} = \sum_{p,n} \frac{1}{np^{2ns}} = \log \zeta(2s).$$
Thus $D(s) \to \infty$ as $s \to 1/2^+$.

The third bullet contradicts the second one, so the supposition $L(1, \chi) = 0$ is untenable.

3. Landau's Lemma

Proposition 3.1 (Weak Version of Landau's Lemma)**.** *Suppose that the Dirichlet series with nonnegative coefficients*
$$D(s) = \sum_{n \geq 1} a_n n^{-s}, \quad a_n \geq 0 \text{ for all } n$$
converges to an analytic function $f(s)$ on the open right half plane $\{\mathrm{Re}(s) > \sigma_o\}$. Suppose that for some $\varepsilon > 0$, the function $f(s)$ extends analytically to the larger open right half plane $\{\mathrm{Re}(s) > s_o - \varepsilon\}$. Then the Dirichlet series $D(s)$ converges to $f(s)$ on the x-axis portion of the larger right half plane, i.e., $D(\sigma) = f(\sigma)$ for all $\sigma \in (\sigma_o - \varepsilon, \sigma_o)$.

Proof. By way of quick review, recall the basic definition
$$a^z = e^{z \log a}, \quad a \in \mathbf{R}^+, \ z \in \mathbf{C},$$
so that the derivatives of a^z are
$$(a^z)^{(k)} = (\log a)^k a^z, \quad a \in \mathbf{R}^+, \ z \in \mathbf{C}, \ k \in \mathbf{Z}_{\geq 0}.$$
Thus the power series expansion of a^z about $z = 0$ is
$$a^z = \sum_{k \geq 0} \frac{(\log a)^k}{k!} z^k, \quad a \in \mathbf{R}^+, \ z \in \mathbf{C},$$
We will refer back to this expansion later in the argument.

Returning to Landau's Lemma, we may translate the problem and take $\sigma_o = 0$. (The translation leaves the Dirichlet series coefficients nonnegative.) The function $f(s)$ is analytic on $B(1, 1 + \varepsilon)$. Thus for any $\sigma \in (-\varepsilon, 0)$ the power series representation of $f(s)$ about $s = 1$ converges at σ to $f(\sigma)$,
$$f(\sigma) = \sum_{k \geq 0} \frac{f^{(k)}(1)}{k!} (\sigma - 1)^k = \sum_{k \geq 0} \frac{(-1)^k f^{(k)}(1)}{k!} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0.$$
Since the Dirichlet series $D(s) = \sum_{n \geq 1} a_n n^{-s}$ converges to $f(s)$ about $s = 1$, compute $(-1)^k f^{(k)}(1)$ by differentiating $D(s)$ termwise,
$$(-1)^k f^{(k)}(1) = (-1)^k \sum_{n \geq 1} a_n (-\log n)^k n^{-s} \Big|_{s=1} = \sum_{n \geq 1} \frac{a_n (\log n)^k}{n}.$$
Thus
$$f(\sigma) = \sum_{k \geq 0} \sum_{n \geq 1} \frac{a_n (\log n)^k}{k! n} (1 - \sigma)^k, \quad -\varepsilon < \sigma < 0.$$

All of the terms are nonnegative, so we may rearrange the sum,

$$f(\sigma) = \sum_{n\geq 1} \frac{a_n}{n} \sum_{k\geq 0} \frac{(\log n)^k}{k!}(1-\sigma)^k, \quad -\varepsilon < \sigma < 0.$$

As explained at the beginning of the proof, the inner sum is the power series expansion of n^s about 0 at $s = 1 - \sigma$. Thus

$$f(\sigma) = \sum_{n\geq 1} \frac{a_n}{n} n^{1-\sigma} = \sum_{n\geq 1} a_n n^{-\sigma} = D(\sigma), \quad -\varepsilon < \sigma < 0.$$

This is the desired result. □