1. Algebraic numbers and algebraic integers

We like numbers such as \( i \) and \( \omega = \zeta_3 = e^{2\pi i/3} \) and \((1 + \sqrt{5})/2\) and so on. To think about such numbers in a structured way is to think of them not as radicals, but as roots.

**Definition 1.1.** A complex number \( \alpha \) is an **algebraic number** if \( \alpha \) satisfies some monic polynomial with rational coefficients,

\[
p(\alpha) = 0, \quad p(x) = x^n + c_1 x^{n-1} + \cdots + c_n, \quad c_1, \ldots, c_n \in \mathbb{Q}.
\]

Every rational number \( r \) is algebraic since it satisfies the polynomial \( p(x) = x - r \), but not every algebraic number is rational, cf. the examples given just before the definition. E.g., \((1 + \sqrt{5})/2\) satisfies the polynomial \( p(x) = x^2 - x - 1 \). Every complex number expressible over \( \mathbb{Q} \) in radicals is algebraic, but not conversely.

The algebraic numbers form a field, denoted \( \overline{\mathbb{Q}} \). This is shown as follows.

**Theorem 1.2.** Let \( \alpha \) be a complex number. The following conditions on \( \alpha \) are equivalent:

1. \( \alpha \) is an algebraic number, i.e., \( \alpha \in \overline{\mathbb{Q}} \),
2. The ring \( \mathbb{Q}[\alpha] \) is a finite-dimensional vector space over \( \mathbb{Q} \),
3. \( \alpha \) belongs to a ring \( R \) in \( \mathbb{C} \) that is finitely generated over \( \mathbb{Q} \).

**Proof.** (1) \( \implies \) (2): Let \( \alpha \) satisfy the monic polynomial \( p(x) \in \mathbb{Q}[x] \), and let \( n = \deg(p) \). For any nonnegative integer \( m \), the division algorithm in \( \mathbb{Q}[x] \) gives

\[
x^m = q(x)p(x) + r(x), \quad \deg(r) < n \text{ or } r = 0.
\]

Thus, since \( p(\alpha) = 0 \),

\[
\alpha^m = r(\alpha) \in \mathbb{Q} \oplus \mathbb{Q}\alpha \oplus \cdots \oplus \mathbb{Q}\alpha^{n-1}.
\]

Since \( \mathbb{Q}[\alpha] \) is generated over \( \mathbb{Q} \) as a vector space by the nonnegative powers of \( \alpha \), this shows that \( \mathbb{Q}[\alpha] \) is generated by the finite set \( \{1, \alpha, \ldots, \alpha^{n-1}\} \).

(2) \( \implies \) (3) is immediate.

(3) \( \implies \) (1): Let the ring \( R \) have nonzero generators \( g_1, \ldots, g_n \) over \( \mathbb{Q} \). Multiplying each generator by \( \alpha \) gives a rational linear combination of the generators,

\[
\alpha g_i = \sum_{j=1}^{n} c_{ij} g_j, \quad i = 1, \ldots, n.
\]

That is, letting \( M = [c_{ij}] \in \mathbb{Q}^{n \times n} \),

\[
\alpha \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = M \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.
\]

This shows that \( \alpha \) is an eigenvalue of \( M \), and so it satisfies the characteristic polynomial of \( M \), a monic polynomial with rational coefficients. \( \square \)
The implications \((1) \implies (2) \implies (3)\) in the theorem are essentially trivial. The one idea in the theorem is the argument that \((3) \implies (1)\) in consequence of \(\alpha\) being an eigenvalue. Here the ring structure and the vector space structure of \(R\) interact. For example, if we take \(\alpha = \omega = \zeta_3 = e^{2\pi i/3}\) and \(R = \mathbb{Q}[\alpha]\) then multiplication by \(\alpha\) takes 1 to \(\alpha\) and \(\alpha\) to \(\alpha^2 = -1 - \alpha\), so the matrix in the proof that \((3) \implies (1)\) is 
\[
M = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},
\]
whose characteristic polynomial \(\det(xI - M) = x^2 + x + 1\) is indeed the characteristic polynomial of \(\alpha\).

Condition (3) in the theorem easily proves

**Corollary 1.3.** The algebraic numbers \(\overline{\mathbb{Q}}\) form a field.

**Proof.** Let \(\alpha\) and \(\beta\) be algebraic numbers. Then the rings \(\mathbb{Q}[\alpha]\) and \(\mathbb{Q}[\beta]\) have respective bases 
\[
\{\alpha^i : 0 \leq i < m\} \quad \text{and} \quad \{\beta^j : 0 \leq j < n\}
\]
as vector spaces over \(\mathbb{Q}\). Let 
\[
R = \mathbb{Q}[\alpha, \beta],
\]
spanned as a vector space over \(\mathbb{Q}\) by the set 
\[
\{\alpha^i\beta^j : 0 \leq i < m, 0 \leq j < n\}.
\]
Then \(\alpha + \beta\) and \(\alpha\beta\) belong to \(R\), making them algebraic numbers by condition (3) of the theorem. If \(\alpha \neq 0\) then its polynomial \(p(x)\) can be taken to have a nonzero constant term \(c_n\) after dividing through by its lowest power of \(x\). The relations \(\alpha | c_n - p(\alpha)\) in \(\mathbb{Q}[\alpha]\) and \(p(\alpha) = 0\) give \(\alpha | c_n\) in \(\mathbb{Q}[\alpha]\), so that 
\[
\alpha^{-1} = (1/c_n) \cdot (c_n/\alpha) \in \mathbb{Q}[\alpha],
\]
making \(\alpha^{-1}\) an algebraic number by condition (3) as well. \(\square\)

If \(\alpha\) and \(\beta\) are algebraic numbers satisfying the monic rational polynomials \(p(x)\) and \(q(x)\) then the proofs of Corollary 1.3 and of \((3) \implies (1)\) in Theorem 1.2 combine to produce the polynomials satisfied by \(\alpha + \beta\) and \(\alpha\beta\) and \(1/\alpha\) if \(\alpha \neq 0\). For example, let \(\alpha = i\) and \(\beta = \sqrt{2}\). Then 
\[
\mathbb{Q}[i, \sqrt{2}] = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}\sqrt{2} \oplus \mathbb{Q}i\sqrt{2}.
\]
Compute that 
\[
(i + \sqrt{2}) \begin{bmatrix} 1 \\ i \\ \sqrt{2} \\ i\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ \sqrt{2} \\ i\sqrt{2} \end{bmatrix}.
\]
Thus \(i + \sqrt{2}\) satisfies the characteristic polynomial of the matrix in the display.

The theory of **resultants** provides a general algorithm to find such polynomials. The idea is that given any field \(k\), and given any two nonzero polynomials \(f(T), g(T) \in k[T]\), their resultant 
\[
R(f(T), g(T)) \in k
\]
is zero if and only if \(f\) and \(g\) share a root that is algebraic over \(k\). That is:

**The condition** \(R(f(T), g(T)) = 0\) **eliminates the variable** \(T\) **from the simultaneous equations** \(f(T) = 0, g(T) = 0\).
Now, suppose that the algebraic numbers $\alpha$ and $\beta$ respectively satisfy the polynomials $f(T)$ and $g(U)$ over $\mathbb{Q}$. Then the condition

$$R(f(T), R(g(U), T + U - V)) = 0$$

first eliminates $U$ from the simultaneous conditions $g(U) = 0$, $T = U + V$, leaving a polynomial condition $h(T, V) = 0$, and then it eliminates $T$ from the simultaneous conditions $f(T) = 0$, $h(T, V) = 0$, leaving a polynomial $k(V)$ over $\mathbb{Q}$ having $\alpha + \beta$ as a root. Almost identically, the condition

$$R(f(T), R(g(U), TU - V)) = 0$$

is a polynomial condition $k(V)$ over $\mathbb{Q}$ having $\alpha\beta$ as a root.

One can now consider complex numbers $\alpha$ satisfying monic polynomials with coefficients in $\mathbb{Q}$. But in fact $\mathbb{Q}$ is algebraically closed, meaning that any such $\alpha$ is already in $\mathbb{Q}$. The proof again uses condition (3) in the theorem.

**Corollary 1.4.** The field $\mathbb{Q}$ of algebraic numbers is algebraically closed.

**Proof.** (Sketch.) Consider a monic polynomial

$$x^n + c_1 x^{n-1} + \cdots + c_n, \quad c_i \in \mathbb{Q},$$

and let $\alpha$ be one of its roots. Since each ring $\mathbb{Q}[c_i]$ is a finite-dimensional vector space over $\mathbb{Q}$, so is the ring

$$R_o = \mathbb{Q}[c_1, \ldots, c_n].$$

Let

$$R = R_o[\alpha].$$

If $\{v_i : 1 \leq i \leq m\}$ is a basis for $R_o$ over $\mathbb{Q}$ then

$$\{v_i \alpha^j : 1 \leq i \leq m, 0 \leq j < n\}$$

is a spanning set for $R$ as a vector space over $\mathbb{Q}$. (This set is not necessarily a basis since $\alpha$ might satisfy a polynomial of lower degree.) Now condition (3) of the theorem shows that $\alpha \in \mathbb{Q}$. $\square$

A loose analogy holds here:

*Just as the finite-cover notion of compactness clarifies analytic phenomena by translating them into topological terms, so finite generation clarifies algebraic phenomena by translating them into structural terms.*

The slogan for the proof that the field of algebraic numbers is algebraically closed is *finitely generated over finitely generated is finitely generated.*

The ring of integers $\mathbb{Z}$ in the rational number field $\mathbb{Q}$ has a natural analogue in the field of algebraic numbers $\mathbb{Q}$. To begin discussing this situation, note that any algebraic number satisfies a *unique* monic polynomial of lowest degree, since subtracting two distinct monic polynomials of the same degree gives a nonzero polynomial of lower degree, which can be rescaled to be monic. The unique monic polynomial of least degree satisfied by an algebra number $\alpha$ is called the **minimal polynomial** of $\alpha$.

**Definition 1.5.** An algebraic number $\alpha$ is an **algebraic integer** if its minimal polynomial has integer coefficients.
The set of algebraic integers is denoted \( \mathbb{Z} \). Immediately from the definition, the algebraic integers in the rational number field \( \mathbb{Q} \) are the usual integers \( \mathbb{Z} \), now called the \textbf{rational integers}. Also in consequence of the definition, a small exercise shows that every algebraic number takes the form of an algebraic integer divided by a rational integer. Note, however, that the algebraic numbers 
\[ \omega = -\frac{1 + i\sqrt{3}}{2} \quad \text{and} \quad \tau = \frac{1 + \sqrt{5}}{2} \]
are algebraic integers despite “having denominators”—indeed, they satisfy the polynomials \( x^2 + x + 1 \) and \( x^2 - x - 1 \) respectively. Similarly to Theorem 1.2 and its corollaries,

\textbf{Theorem 1.6.} Let \( \alpha \) be a complex number. The following conditions on \( \alpha \) are equivalent:

1. \( \alpha \) is an algebraic integer, i.e., \( \alpha \in \mathbb{Z} \),
2. The ring \( \mathbb{Z}[\alpha] \) is finitely generated as an Abelian group,
3. \( \alpha \) belongs to a ring \( \mathcal{R} \) in \( \mathbb{C} \) that is finitely generated as an Abelian group.

\textbf{Corollary 1.7.} The algebraic integers \( \mathbb{Z} \) form a ring.

\textbf{Corollary 1.8.} The algebraic integers form an \textbf{integrally closed} ring, meaning that every monic polynomial with coefficients in \( \mathbb{Z} \) factors down to linear terms over \( \mathbb{Z} \), i.e., its roots lie in \( \mathbb{Z} \).

2. \textbf{Quadratic Reciprocity Revisited}

We work in the ring \( \mathbb{Z} \) of algebraic integers, remembering at the end that an algebraic integer congruence between two rational integers is in fact a rational integer congruence. (Proof: If 
\[ a, b \in \mathbb{Z} \quad \text{and} \quad a = b \mod p \mathbb{Z}, \]
then 
\[ \frac{b - a}{p} \in \mathbb{Q} \cap \mathbb{Z} = \mathbb{Z}, \]
so that 
\[ a = b \mod p \mathbb{Z}, \]
as desired.)

Let \( p \) be an odd prime. To evaluate the Legendre symbol \( (2/p) \), introduce not the square root of unity but the eighth root of unity,
\[ \zeta = \zeta_8 = e^{2\pi i/8} \in \mathbb{Z}. \]

Since \( \zeta^4 = -1 \), consequently \( \zeta^2 + \zeta^{-2} = 0 \), and thus \( (\zeta + \zeta^{-1})^2 = 2 \). Also, a small calculation shows that working modulo \( p \mathbb{Z} \),
\[ (\zeta + \zeta^{-1})^p = \begin{cases} \zeta + \zeta^{-1} & \text{if } p = \pm 1 \mod 8, \\ -(\zeta + \zeta^{-1}) & \text{if } p = \pm 3 \mod 8. \end{cases} \]
\[ = (\zeta + \zeta^{-1})(-1)^{(p^2-1)/8}. \]

Let 
\[ \tau = \zeta + \zeta^{-1}, \]
and compute \( \tau^{p+1} \) in two different ways. First, using Euler’s law at the last step,

\[
\tau^{p+1} = \tau^2(\tau^2)^{(p-1)/2} = 2 \cdot 2^{(p-1)/2} \equiv 2 \left( \frac{2}{p} \right).
\]

And second, quoting the small calculation,

\[
\tau^{p+1} = \tau \cdot \tau^p \equiv \tau^2(-1)^{(p^2-1)/8} = 2(-1)^{(p^2-1)/8}.
\]

Thus we have a congruence in \( \mathbb{Z} \),

\[
2 \left( \frac{2}{p} \right) \equiv 2(-1)^{(p^2-1)/8} \pmod{p\mathbb{Z}},
\]

but since both quantities are rational integers we may view the congruence as being set in \( \mathbb{Z} \). Since \( p \) is odd, we may cancel the 2’s,

\[
\left( \frac{2}{p} \right) \equiv (-1)^{(p^2-1)/8} \pmod{p\mathbb{Z}},
\]

and again since \( p \) is odd and since the integers on each side of the congruence are \( \pm 1 \), the integers must be equal,

\[
\left( \frac{2}{p} \right) = (-1)^{(p^2-1)/8}.
\]

The proof of the main quadratic reciprocity law is similar. Let \( p \) and \( q \) be odd primes. (In this argument, \( p \) and \( q \) will play roles respectively analogous to those played by 2 and \( p \) a moment ago.) Introduce the \( p \)th root of unity,

\[
\zeta = \zeta_p = e^{2\pi i/p}.
\]

The finite geometric sum formula gives

\[
\sum_{a=1}^{p-1} \zeta^{at} = \begin{cases} p - 1 & \text{if } t \equiv 0 \pmod{p}, \\ -1 & \text{if } t \not\equiv 0 \pmod{p}. \end{cases}
\]

Define the Gauss sum

\[
\tau = \sum_{t=1}^{p-1} \left( \frac{t}{p} \right) \zeta^t.
\]

(Yes, the Gauss sum seems to come out of nowhere. In fact it is a very easy case of a Lagrange resolvent.) Compute that the Gauss sum lets us express \( p \) in terms of \( p \)th roots of unity,

\[
\tau^2 = \sum_{s,t} \left( \frac{s}{p} \right) \left( \frac{t}{p} \right) \zeta^{s+t} = \sum_{s,u} \left( \frac{s}{p} \right) \left( \frac{su}{p} \right) \zeta^{s(1+u)} = \sum_{u} \left( \frac{u}{p} \right) \sum_{s} \zeta^{s(1+u)}
\]

\[
= - \sum_{u \not\equiv -1} \left( \frac{u}{p} \right) + \left( \frac{-1}{p} \right)(p-1) = - \sum_{u} \left( \frac{u}{p} \right) + \left( \frac{-1}{p} \right)(p-1)
\]

\[
= p^*, \quad \text{where } p^* \text{ denotes whichever of } \pm p \text{ is } 1 \pmod{4}.
\]

Now similarly to above, we compute \( \tau^{q+1} \) in two ways. First, by Euler’s Law,

\[
\tau^{q+1} = \tau^2(\tau^2)^{(q-1)/2} = p^*(p^*)^{(q-1)/2} \equiv p^* \left( \frac{p^*}{q} \right).
\]
And second,
\[ \tau q^{q+1} = \tau \cdot \tau q \equiv \tau \sum_t \left( \frac{t}{p} \right) q^{t} = \tau \left( \frac{q}{p} \right) \sum_t \left( \frac{qt}{p} \right) q^{t} = \tau^2 \left( \frac{q}{p} \right) = p^* \left( \frac{q}{p} \right). \]
Thus we have a congruence in \( \mathbb{Z} \),
\[ p^* \left( \frac{p^*}{q} \right) = p^* \left( \frac{q}{p} \right) \mod q\mathbb{Z}, \]
but since both quantities are rational integers we may view the congruence as being set in \( \mathbb{Z} \). Since \( p \) and \( q \) are distinct, we may cancel the \( p^* \)'s,
\[ \left( \frac{p^*}{q} \right) = \left( \frac{q}{p} \right) \mod q\mathbb{Z}, \]
and since \( q \) is odd and since the integers on each side of the congruence are \( \pm 1 \), the integers must be equal,
\[ \left( \frac{p^*}{q} \right) = \left( \frac{q}{p} \right). \]

3. Sketch of a Modern Proof of Quadratic Reciprocity

Let \( p \) be an odd prime, let \( \zeta = e^{2\pi i/p} \), and consider the cyclotomic number field
\[ K = \mathbb{Q}(\zeta). \]
Its Galois group
\[ G = \text{Gal}(K/\mathbb{Q}) \]
is naturally isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^\times \), the automorphism that takes \( \zeta \) to \( \zeta^m \) mapping to the residue class \( m \mod p \) for \( m \in \{1, \ldots, p-1\} \).
Since the Galois group is cyclic of even order, the cyclotomic field \( K \) has a unique quadratic subfield. To describe this field, let \( p^* = (-1)^{(p-1)/2}/p \); thus \( p^* \) is whichever of \( \pm p \) equals 1 mod 4. We know that \( p^* \) is a square in \( K \), the square of the Gauss sum \( \tau \). Consequently, the unique quadratic subfield of \( K \) is
\[ F = \mathbb{Q}(\sqrt{p^*}). \]
Its Galois group
\[ Q = \text{Gal}(F/\mathbb{Q}) \]
is naturally isomorphic to \( \{ \pm 1 \} \), the nontrivial automorphism that takes \( \sqrt{p^*} \) to \( -\sqrt{p^*} \) mapping to \( -1 \). Summarizing so far, we have a commutative diagram in which the horizontal arrows are isomorphisms, so that the left vertical arrow (restriction of automorphisms) gives rise to the right vertical arrow,
\[ \begin{array}{ccc}
G & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^\times \\
\downarrow & & \downarrow \\
Q & \longrightarrow & \{ \pm 1 \}.
\end{array} \]
Any odd prime \( q \neq p \) is unramified in \( K \), and so it has a unique Frobenius automorphism,
\[ \text{Frob}_{q,K} \in G, \]
whose action on the integers of \( K \) is characterized by the condition
\[ x^{\text{Frob}_{q,K}} = x^q \mod q\mathcal{O}_K, \quad x \in \mathcal{O}_K. \]
The Frobenius automorphism has no choice but to be
\[ \text{Frob}_{q,K} : \zeta \mapsto \zeta^q. \]
The odd prime \( q \neq p \) is also unramified in \( F \), so that again it has a unique Frobenius automorphism, this time denoted
\[ \text{Frob}_{q,F} \in \mathbb{Q}, \]
characterized by the condition
\[ x^{\text{Frob}_{q,F}} = x^q \mod q\mathcal{O}_F, \quad x \in \mathcal{O}_F, \]
and (since \( (p^*)^{(q-1)/2} = (p^* / q) \mod q \), where \( (p^* / q) \) is the Legendre symbol) working out to
\[ \text{Frob}_{q,F} : \sqrt{p} \mapsto (p^* / q) \sqrt{p}. \]
Finally, \( \text{Frob}_{q,F} \) is the restriction of \( \text{Frob}_{q,K} \) to \( F \). So, in the commutative diagram
\[ \begin{array}{ccc}
\text{Frob}_{q,K} & \rightarrow & q \mod p \\
\downarrow & & \downarrow \\
\text{Frob}_{q,F} & \rightarrow & (p^* / q).
\end{array} \]
The right vertical arrow shows that:
\[ \text{As a function of } q, \ (p^* / q) \ \text{depends only on } q \mod p. \]
This is quadratic reciprocity. (A variant of the punchline is that the map down the right side is \( q \mod p \mapsto (q/p) \), and so the commutative diagram shows that \( (p^* / q) = (q/p) \).)

4. The Sign of the Quadratic Gauss Sum

Let \( p \) be an odd prime and let \( p^* = (-1)^{(p-1)/2} p \); thus \( p^* \) is whichever of \( \pm p \) equals 1 mod 4. Let \( \zeta = e^{2\pi i/p} \) and let \( \tau \) denote the quadratic Gauss sum modulo \( p \),
\[ \tau = \sum_{t=1}^{p-1} (t/p) \zeta^t. \]
We know that \( \tau^2 = p^* \), so that
\[ \tau = \begin{cases} 
\pm \sqrt{p} & \text{if } p = 1 \pmod{4}, \\
\pm i \sqrt{p} & \text{if } p = 3 \pmod{4}.
\end{cases} \]
We now show, following Ireland and Rosen, that in both cases the sign is “+”. The proof establishes first that a certain product equals \( \sqrt{p} \) if \( p = 1 \) mod 4 and equals \( i \sqrt{p} \) if \( p = 3 \) mod 4, and then it establishes that the product also equals the Gauss sum.

It is elementary that
\[ \sum_{j=0}^{p-1} X^j = \prod_{j=1}^{p-1} (X - \zeta^j) = \prod_{j=1}^{(p-1)/2} (X - \zeta^j)(X - \zeta^{-j}). \]
But the product as written is overspecific in that the exponents of \( \zeta \) need only to vary through any set of nonzero residue classes modulo \( p \). The residue system that
will help us here is the length-(p − 1) arithmetic progression of 2 (mod 4) numbers symmetrized about 0,
\[ \pm(4 \cdot 1 - 2), \pm(4 \cdot 2 - 2), \pm(4 \cdot 3 - 2), \ldots, \pm(4 \cdot \frac{p-1}{2} - 2). \]

Thus
\[ \sum_{j=0}^{p-1} X^j = \prod_{j=1}^{(p-1)/2} (X - \zeta^{4j-2})(X - \zeta^{-(4j-2)}). \]

Substitute \( X = 1 \) to get
\[ p = \prod_{j=1}^{(p-1)/2} (1 - \zeta^{4j-2})(1 - \zeta^{-(4j-2)}) \]
\[ = \prod_{j=1}^{(p-1)/2} (\zeta^{-(2j-1)} - \zeta^{2j-1})(\zeta^{2j-1} - \zeta^{-(2j-1)}), \]
and so multiplying by \((-1)^{(p-1)/2}\) gives
\[ p^* = \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)})^2. \]

It follows that
\[ \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) = \begin{cases} \pm \sqrt{p} & \text{if } p = 1 \pmod{4}, \\ \pm i \sqrt{p} & \text{if } p = 3 \pmod{4}. \end{cases} \]

The \( j \)th multiplicand is
\[ \zeta^{2j-1} - \zeta^{-(2j-1)} = 2i \sin(2\pi(2j - 1)/p), \]
and the sine is positive for 0 < 2(2j - 1)/p < 1, or 1/2 < j < p/4 + 1/2, or 1 ≤ j < p/4 + 1/2, and similarly the sine is negative for p/4 + 1/2 < j ≤ (p − 1)/2.

If \( p = 1 \pmod{4} \) then the sine is positive for \( j = 1, \ldots, (p - 1)/4 \), and so
\[ \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) = (\text{positive number}) \times i^{(p-1)/2} (-1)^{(p-1)/2 - (p-1)/4} \]
\[ = (\text{positive number}) \times (-1)^{(p-1)/4} (-1)^{(p-1)/4}, \]
a positive number. If \( p = 3 \pmod{4} \) then the sine is positive for \( j = 1, \ldots, (p+1)/4 \), and so
\[ \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) = (\text{positive number}) \times i^{(p-1)/2} (-1)^{(p-1)/2 - (p+1)/4} \]
\[ = (\text{positive number}) \times i(-1)^{(p-3)/4} (-1)^{(p-3)/4}, \]
a positive multiple of \( i \). Thus both “±” signs are positive,
\[ \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) = \begin{cases} \sqrt{p} & \text{if } p = 1 \pmod{4}, \\ i\sqrt{p} & \text{if } p = 3 \pmod{4}. \end{cases} \]

To complete the argument, we need to show that the Gauss sum \( \tau \) equals the product \( \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) \) rather than its negative.
Let 

\[ \tau = \varepsilon \prod_{j=1}^{(p-1)/2} (\zeta^{2j-1} - \zeta^{-(2j-1)}) , \]

where we know that \( \varepsilon = \pm 1 \) and we want to show that \( \varepsilon = 1 \). Consider the polynomial

\[ f(X) = \sum_{t=1}^{p-1} \frac{(t/p)X^t}{(p-1)!} - \varepsilon \prod_{j=1}^{(p-1)/2} (X^{2j-1} - X^{p-(2j-1)}) . \]

Then \( f(1) = 0 \) and \( f(\zeta) = 0 \). So \( f(X) \) is divisible by \( X^p - 1 \),

\[ \sum_{t=1}^{p-1} \frac{(t/p)X^t}{(p-1)!} - \varepsilon \prod_{j=1}^{(p-1)/2} (X^{2j-1} - X^{p-(2j-1)}) = (X^p - 1)g(X) . \]

Replace \( X \) by \( e^z \) to get

\[ (1) \sum_{t=1}^{p-1} \frac{(t/p)e^{tz}}{(p-1)!} - \varepsilon \prod_{j=1}^{(p-1)/2} (e^{(2j-1)z} - e^{(p-(2j-1))z}) = (e^{pz} - 1)g(e^z) . \]

On the left side of (1), each multiplicand has constant term 0, so that the lowest exponent of \( z \) in the product is \( (p-1)/2 \), and each multiplicand has linear term \( (4j - p - 2)z \). Thus the overall coefficient of \( z^{(p-1)/2} \) on the left side of (1) is

\[ \sum_{t=1}^{(p-1)/2} \frac{(t/p)e^{tz}}{(p-1)!} - \varepsilon \prod_{j=1}^{(p-1)/2} (4j - p - 2) . \]

On the right side of (1), each coefficient of the power series expansion

\[ e^{pz} - 1 = \sum_{n=1}^{\infty} \frac{p^n}{n!} z^n \]

has more powers of \( p \) in its numerator than in its denominator. Thus the coefficient of \( z^{(p-1)/2} \) on the left side of (1) is 0 modulo \( p \), and so after clearing a denominator we have

\[ \sum_{t=1}^{p-1} \frac{(t/p)e^{tz}}{(p-1)!} = \varepsilon((p-1)/2)! \prod_{j=1}^{(p-1)/2} (4j - 2) . \]

Working modulo \( p \), and quoting Euler’s Law and then Fermat’s Little Theorem, the left side is

\[ \sum_{t=1}^{p-1} \frac{(t/p)e^{tz}}{(p-1)!} = \sum_{t=1}^{(p-1)/2} \frac{t^{(p-1)/2}}{t^{(p-1)/2}} e^{t(4j-2)} = \sum_{t=1}^{(p-1)/2} 1 = 1 = -1 . \]

and the right side is

\[ \varepsilon((p-1)/2)! \prod_{j=1}^{(p-1)/2} (4j - 2) = \varepsilon((p-1)/2)!2^{(p-1)/2} \prod_{j=1}^{(p-1)/2} (2j - 1) = \varepsilon(2 \cdot 4 \cdots (p-1))(1 \cdot 3 \cdots (p-2)) = \varepsilon(p-1)! = -\varepsilon \]

by Wilson’s Theorem.

So \( \varepsilon = 1 \) and the argument is complete.
5. The Sign of the Quadratic Gauss Sum by Fourier Analysis

Let $p$ be an odd prime and let $p^* = (-1)^{(p-1)/2}p$; thus $p^*$ is whichever of $\pm p$ equals 1 mod 4. Let $e_p(x) = e^{2\pi ix/p}$ for $x \in \mathbb{R}$, and let $\tau$ denote the quadratic Gauss sum modulo $p$,

$$\tau = \sum_{t=1}^{p-1} (t/p)e_p(t).$$

We know that $\tau^2 = p^*$, so that

$$\tau = \begin{cases} \pm \sqrt{p} & \text{if } p = 1 \pmod{4}, \\ \pm i\sqrt{p} & \text{if } p = 3 \pmod{4}. \end{cases}$$

We show again, using Fourier analysis, that in both cases the sign is “+”.

Begin with the observations that (letting $R$ stand for residue and $N$ for non-residue)

$$\tau = \sum_{R} e_p(R) - \sum_{N} e_p(N),$$

while by the finite geometric sum formula

$$0 = 1 + \sum_{R} e_p(R) + \sum_{N} e_p(N),$$

so that adding the previous two displays shows that the Gauss sum is

$$\tau = 1 + 2 \sum_{R} e_p(R) = \sum_{n=0}^{p-1} e^{2\pi in^2/p}.$$

We will evaluate the more general sum associated to any positive integer $N$,

$$S_N = \sum_{n=0}^{N-1} e^{2\pi in^2/N}, \quad N \in \mathbb{Z}_{>0}.$$ 

The result, encompassing the sign of the Gauss sum, will be that

$$S_N = \sqrt{N} \cdot \begin{cases} 1 + i & \text{if } N = 0 \pmod{4}, \\ 1 & \text{if } N = 1 \pmod{4}, \\ 0 & \text{if } N = 2 \pmod{4}, \\ i & \text{if } N = 3 \pmod{4}. \end{cases}$$

Define

$$f : [0, 1] \to \mathbb{C}, \quad f(x) = e^{2\pi iNx^2}.$$ 

As explained below, the Fourier series of $f$ converges pointwise to $f$ even at 0 because $f(1) = f(0)$ and $f$ is differentiable from the right at 0 and from the left at 1. Thus

$$S_N = \sum_{n=0}^{N-1} f(n/N) = \sum_{n=0}^{N-1} \sum_{k \in \mathbb{Z}} \int_0^1 f(x)e^{2\pi ikx} \, dx \cdot e^{-2\pi ikn/N}$$

$$= \sum_{k \in \mathbb{Z}} \int_0^1 f(x)e^{2\pi ikx} \, dx \cdot \sum_{n=0}^{N-1} e^{-2\pi ikn/N}. $$
The inner sum is \( N \) if \( k \in N \mathbb{Z} \) and 0 otherwise, so now
\[
S_N = \sum_{k \in \mathbb{Z}} N \int_0^1 f(x)e^{2\pi i Nkx} \, dx.
\]

To analyze the \( k \)th summand, complete the square in the exponent and then note that \( k^2 \) is 0 mod 4 for \( k \) even and is 1 mod 4 for \( k \) odd,
\[
N \int_0^1 f(x)e^{2\pi i Nkx} \, dx = N \int_0^1 e^{2\pi i N(x^2 + kx)} \, dx = Ne^{-2\pi i Nk^2/4} \int_0^{k/2} e^{2\pi i Nk^2} \, dx = N \int_{k/2}^{k/2+1} e^{2\pi i Nk^2} \, dx \cdot \begin{cases} 1 & \text{if } k = 0 \text{ mod 2} \\ i^{-N} & \text{if } k = 1 \text{ mod 2} \end{cases}.
\]

As \( k \) varies through the even integers, the last integral in the previous display runs over \( \mathbb{R} \), and similarly for the odd integers. Thus the summing over the last expression in the previous display gives
\[
S_N = N(1 + i^{-N}) \int_{\mathbb{R}} e^{2\pi i Nx^2} \, dx = \sqrt{N}(1 + i^{-N}) \int_{\mathbb{R}} e^{2\pi i x^2} \, dx.
\]
The integral in the previous display converges, as can be seen by a change of variable and then integration by parts: for \( 0 < L \leq M \),
\[
\int_L^M e^{2\pi i x^2} \, dx = \frac{1}{2} \int_L^M x^{-1/2}e^{2\pi i x} \, dx = \frac{1}{4\pi i} \left( x^{-1/2} e^{2\pi i x} \bigg|_L^M + \frac{1}{2} \int_L^M x^{-3/2} e^{2\pi i x} \, dx \right),
\]
which is asymptotically \( L^{-1} \). In the formula \( S_N = \sqrt{N}(1 + i^{-N}) \int_{\mathbb{R}} e^{2\pi i x^2} \, dx \), set \( N = 1 \) to get \( \int_{\mathbb{R}} e^{2\pi i x^2} \, dx = (1 + i)/2 \). Thus the value of \( S_N \) is
\[
S_N = \sqrt{N} \frac{(1 + i^{-N})(1 + i)}{2}.
\]
The casewise formula for \( S_N \) follows immediately.

Finally, to show the pointwise convergence of the Fourier series of \( f \) to \( f \) at 0, replace \( f \) by \( f - f(0) \) and compute the \( M \)th partial sum of the Fourier series at 0,
\[
\sum_{k=-M}^M \int_0^1 f(x)e^{2\pi ikx} \, dx = \int_0^1 f(x) \sum_{k=-M}^M e^{2\pi ikx} \, dx = \int_0^1 \frac{f(x)}{e^{2\pi ix} - 1} (e^{2\pi i(M+1)x} - e^{-2\pi iMx}) \, dx.
\]
The term \( f(x)/(e^{2\pi ix} - 1) = f(x)/x \cdot x/(e^{2\pi ix} - 1) \) in the integrand extends continuously at \( x = 0 \) to the right derivative \( f'(0) \) times the reciprocal derivative \( 1/(2\pi i) \) of \( e^{2\pi ix} \) at 0, and similarly it extends continuously at \( x = 1 \). Thus the integral goes to 0 as \( M \) grows, by the Riemann–Lebesgue Lemma.