1. Introduction

Everybody knows that three hours after 10:00, the time is 1:00. That is, everybody is familiar with modular arithmetic, the usual arithmetic of the integers subject to the additional condition that some fixed integer (such as 12) is treated as 0. After noting some quick consequences of the Euclidean structure of the integer ring \((\mathbb{Z}, +, \cdot)\), this lecture places modular arithmetic in the context of quotient structures of this ring.

2. Some Loose Ends

2.1. Euclid’s Lemma. Euclid’s Lemma states that for positive integers \(a, b, n\):

\[
\text{If } n \mid ab \text{ and } \gcd(n, a) = 1 \text{ then } n \mid b.
\]

Note that Euclid’s Lemma is similar to the definition of a prime element of an integral domain: both have the premise of an element dividing a product, and then the conclusion that the element divides one of the multiplicands. And, indeed, the proof of Euclid’s Lemma is essentially the same as the proof that each irreducible element is prime in a Euclidean domain. Specifically, we have \(n \mid ab\) and, because \(\gcd(n, a) = 1\), we also have \(Nn + Aa = 1\) for some \(N, A\). Multiply this relation through by \(b\) to get \(Nnb + Aab = b\). But \(n\) divides the left side, so \(n\) divides the right side, as desired.

We can also state and prove Euclid’s Lemma in the language of ideals, notwithstanding that doing so is anti-historic. Again for positive integers \(a, b, n\), now the statement is:

\[
\text{If } (ab) \subset (n) \text{ and } (n, a) = (1) \text{ then } (b) \subset (n).
\]

Here the argument is that \((b) = (nb, ab) \subset (nb, n) = (n)\).

A consequence of Euclid’s Lemma is:

\[
\text{If } (a, b) = 1 \text{ then } (a, bc) = (a, c).
\]

Indeed, given that \((a, b) = 1\), for any given \(d\) the conditions \(d \mid a, bc\) further give \((d, b) = 1\), and so Euclid’s Lemma gives \(d \mid c\). Conversely, it is immediate that \(d \mid a, c\) gives \(d \mid bc\). A particular instance of the consequence of Euclid’s Lemma is:

\[
\text{The set of positive integers coprime to a given } n \text{ is closed under multiplication.}
\]

2.2. Least common multiple. Any two positive integers have a positive integer least common multiple, denoted \(\text{lcm}(a, b)\). Letting \(\ell\) denote this least common multiple, we show that \(ab\) is a multiple of \(\ell\), making \(\ell\) a quotient of \(ab\). The idea is that the division theorem gives

\[
ab = q\ell + r, \quad 0 \leq r < \ell,
\]

from which \(a\) and \(b\) both divide \(r\). But because \(r < \ell\) and \(\ell\) is the least common multiple, \(r = 0\). This gives the result.
Knowing that \( \text{lcm}(a, b) \) is a quotient of \( ab \), we now reason that for any positive integer \( h \), letting \( g = \gcd(a, b) \),

\[
\begin{align*}
\frac{ab}{h} &\iff \frac{ah}{g} \leftarrow \frac{bh}{g} \iff \frac{g}{h} \leftarrow \frac{ab}{h}.
\end{align*}
\]

Thus \( \text{lcm}(a, b) = ab/ \gcd(a, b) \).

This argument demonstrates global methods, as compared to the local approach of factoring \( a \) and \( b \) uniquely into prime powers—as we can do in consequence of a result whose proof is essentially the proof of Euclid’s Lemma—and working a prime at a time. For the local argument, if the relevant powers of a given prime \( p \) in \( a \) and \( b \) are \( p^{e_a}, p^{e_b} \), then the relevant powers of \( p \) in \( ab, \gcd(a, b), \text{lcm}(a, b) \) are \( p^{\max(e_a, e_b)} \). Because \( e_a + e_b = \min(e_a, e_b) + \max(e_a, e_b) \), we have \( ab = \gcd(a, b) \text{lcm}(a, b) \).

Similarly, it is easy to show the consequence of Euclid’s Lemma above by local methods, taking unique factorization as morally in hand once Euclid’s Lemma is proved.

There are tradeoffs between local and global methods, depending on context. One issue in algorithmic/computational number theory is that factorization into primes can be intractable for large integers.

### 3. The Quotient Ring \( \mathbb{Z}/n\mathbb{Z} \)

Let \( n \in \mathbb{Z}^+ \) be a positive integer. Equality up to multiples of \( n \) partitions \( \mathbb{Z} \) into \( n \) equivalence classes, called cosets,

\[
\overline{0} = 0 + n\mathbb{Z}, \quad \overline{1} = 1 + n\mathbb{Z}, \quad \ldots, \quad \overline{n-1} = (n-1) + n\mathbb{Z}.
\]

Let \( \mathbb{Z}/n\mathbb{Z} \) denote the set of these cosets. The map

\[
\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}, \quad a \mapsto \overline{a} = a + n\mathbb{Z}
\]

is a well defined surjection. However, for now its domain is a ring but its codomain is only a set. We want it to be a ring-to-ring map, and this requires addition and multiplication in \( \mathbb{Z}/n\mathbb{Z} \).

The natural addition and multiplication of \( \mathbb{Z}/n\mathbb{Z} \) are obvious:

\[
\overline{a} + \overline{b} \overset{\text{def}}{=} \overline{a + b}, \quad \overline{a} \cdot \overline{b} \overset{\text{def}}{=} \overline{ab}.
\]

Informally we are doing remainder arithmetic, but really an equivalence class such as \( \overline{3} \) means 3 and all its \( n \)-translates. The number of hours from any 10:00 to any 1:00 is \( \overline{3} \) rather than 3. The question is not what the operations of \( \mathbb{Z}/n\mathbb{Z} \) must be, but whether what they must be even makes sense. To address this question, we move to coset notation,

\[
(a + n\mathbb{Z}) + (b + n\mathbb{Z}) \overset{\text{def}}{=} (a + b) + n\mathbb{Z}, \quad (a + n\mathbb{Z})(b + n\mathbb{Z}) \overset{\text{def}}{=} ab + n\mathbb{Z}.
\]

The point is that conceivably \( a + n\mathbb{Z} = a' + n\mathbb{Z} \) and \( b + n\mathbb{Z} = b' + n\mathbb{Z} \) in \( \mathbb{Z}/n\mathbb{Z} \) with \( a \neq a' \) and/or \( b \neq b' \) in \( \mathbb{Z} \), and so the sum \( (a + n\mathbb{Z}) + (b + n\mathbb{Z}) = (a' + n\mathbb{Z}) + (b' + n\mathbb{Z}) \) is defined by two different formulas,

\[
(a + n\mathbb{Z}) + (b + n\mathbb{Z}) = a + b + n\mathbb{Z}, \quad (a' + n\mathbb{Z}) + (b' + n\mathbb{Z}) = a' + b' + n\mathbb{Z}.
\]

Unless \( a + b + n\mathbb{Z} = a' + b' + n\mathbb{Z} \), addition in \( \mathbb{Z}/n\mathbb{Z} \) isn’t sensible. And similarly for multiplication in \( \mathbb{Z}/n\mathbb{Z} \),

\[
(a + n\mathbb{Z})(b + n\mathbb{Z}) = ab + n\mathbb{Z}, \quad (a' + n\mathbb{Z})(b' + n\mathbb{Z}) = a'b' + n\mathbb{Z}.
\]
so unless \(ab + n\mathbb{Z} = a'b' + n\mathbb{Z}\), multiplication in \(\mathbb{Z}/n\mathbb{Z}\) isn’t sensible. However, the conditions

\[
a + n\mathbb{Z} = a' + n\mathbb{Z} \quad \text{and} \quad b + n\mathbb{Z} = b' + n\mathbb{Z}
\]

are

\[
a' - a \in n\mathbb{Z} \quad \text{and} \quad b' - b \in n\mathbb{Z},
\]

which give, crucially using the ideal properties of \(n\mathbb{Z}\) in \(\mathbb{Z}\) as compared to merely its subring properties,

\[
(a' + b') - (a + b) = (a' - a) + (b' - b) \in n\mathbb{Z}
\]

\[
a'b' - ab = a'(b' - b) + b(a' - a) \in n\mathbb{Z},
\]

and these conditions are the desired ones,

\[
a + b + n\mathbb{Z} = a' + b' + n\mathbb{Z} \quad \text{and} \quad ab + n\mathbb{Z} = a'b' + n\mathbb{Z}.
\]

That is, the quotient space \(\mathbb{Z}/n\mathbb{Z}\) inherits ring structure from \(\mathbb{Z}\) because the subring \(n\mathbb{Z}\) of \(\mathbb{Z}\) is an ideal. With the natural candidate inherited addition and multiplication operations of \(\mathbb{Z}/n\mathbb{Z}\) confirmed as sensible, the reduction map

\[
\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}
\]

is innately a surjective ring homomorphism.

With these issues clearly addressed, from now on we allow ourselves to be situationally casual with the notations \(a, a', a + n\mathbb{Z}\). For example, “\(a \in \mathbb{Z}/n\mathbb{Z}\)” and “\(\bar{a} \in \mathbb{Z}/n\mathbb{Z}\) where \(a \in \mathbb{Z}\)” are both literally correct, each assigning a different meaning to \(a\)—especially, \(a\) is not an integer in the first—but we may blur this distinction and let \(a\) denote both quantities simultaneously.

The unit group of \(\mathbb{Z}/n\mathbb{Z}\) is

\[
(\mathbb{Z}/n\mathbb{Z})^\times = \{a \in \mathbb{Z}/n\mathbb{Z} : ab = 1 \text{ for some } b \in \mathbb{Z}/n\mathbb{Z}\}
\]

Our earlier discussion of ideals and the Euclidean algorithm shows that for any integer \(a \in \mathbb{Z}\),

\[
\bar{a} \in (\mathbb{Z}/n\mathbb{Z})^\times \iff \bar{a}\bar{b} = \bar{1} \text{ for some } \bar{b}
\]

\[
\iff ab + kn = 1 \text{ for some } b \text{ and } k \iff (a, n) = 1.
\]

Consequently,

\[
(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} : (a, n) = 1\}.
\]

Here we can note that for any integer \(k\) we have \((a + kn, n) = (a, n)\), so the value of \((a, n)\) is independent of which \(a\) we use to name the coset \(\bar{a}\). Thus the previous display is sensible. It shows that the elementary definition of the Euler totient function,

\[
\phi(n) = |\{a \in \{0,1,\ldots,n-1\} : (a, n) = 1\}|
\]

is equivalent to a more conceptual definition,

\[
\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|.
\]
4. Maps Among Quotient Rings of $\mathbb{Z}$

Let $n$ and $m$ be positive integers with $n \mid m$. The inclusion map $m\mathbb{Z} \to n\mathbb{Z}$ of ideals of $\mathbb{Z}$ gives rise to a surjective map of quotient rings in the other direction,

$$\pi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \quad a + m\mathbb{Z} \mapsto a + n\mathbb{Z}.$$  

Similarly the quotient ring operations, this quotient map is the obvious thing, but what needs to be shown is that thanks to the containment $m\mathbb{Z} \subset n\mathbb{Z}$, it makes sense. The problem is that conceivably an element $a + m\mathbb{Z}$ of $\mathbb{Z}/m\mathbb{Z}$ is also $a' + m\mathbb{Z}$ but $a + n\mathbb{Z}$ and $a' + n\mathbb{Z}$ are distinct in $\mathbb{Z}/n\mathbb{Z}$. However, this can’t happen: the condition $a + m\mathbb{Z} = a' + m\mathbb{Z}$ implies $a' - a \in m\mathbb{Z}$ because of the containment, and this is the condition $a + n\mathbb{Z} = a' + n\mathbb{Z}$.

The surjective map between quotient rings gives rise to a corresponding surjective map of multiplicative groups,$$
\pi : (\mathbb{Z}/m\mathbb{Z})^\times \to (\mathbb{Z}/n\mathbb{Z})^\times, \quad a + m\mathbb{Z} \mapsto a + n\mathbb{Z},
$$
noting that if $(a,m) = 1$ then also $(a,n) = 1$, so indeed the quotient ring map takes units to units. The nonobvious point is that the unit group map surjects. In general, a surjection of commutative rings with 1 needn’t give rise to a surjection of the unit groups, as shown by the map $\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$, so the argument that our map here surjects must use specifics of the situation. The problem is that the condition $(a,n) = 1$ doesn’t imply $(a,m) = 1$. To address this, it suffices to consider the case $m = np$ with $p$ prime, because the general case can be built from this one in finitely many steps. By the consequence of Euclid’s Lemma noted early in this writeup, $(a,m) = (a,p)$. Now there are two cases.

- If $p \nmid a$ then $(a,m) = 1$. Thus $a + m\mathbb{Z}$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ maps to $a + n\mathbb{Z}$ in $(\mathbb{Z}/n\mathbb{Z})^\times$.

- If $p \mid a$ then $(n,p) = 1$ because $(a,n) = 1$, and so $(a + n,p) = 1$. This gives $(a + n,m) = (a + n,n)$ by the consequence of Euclid’s Lemma, and then $(a + n,n) = (a,n) = 1$, so altogether $(a + n,m) = 1$. Thus $a + n + m\mathbb{Z}$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ maps to $a + n\mathbb{Z}$ in $(\mathbb{Z}/n\mathbb{Z})^\times$.

5. Congruence

Definition 5.1. For any integers $a$, $b$, and $n$, we say that $a$ equals $b$ modulo $n$, notated

$$a = b \mod n,$$

if $n \mid b - a$. Other notations for congruence are

$$a \equiv b \pmod{n}, \quad a = b \ (n),$$

and so on.

We recognize congruence modulo $n$ in $\mathbb{Z}$ to mean equality in the quotient ring $\mathbb{Z}/n\mathbb{Z}$. Although there is nothing new in the definition other than notation, the notation lets us phrase arguments neatly and naturally. Here are some examples.

- $a = b \mod 0$ if and only if $a = b$.
- $a = b \mod 1$ for all $a$ and $b$.
- $a = b \mod 2$ if and only if $a$ and $b$ have the same parity.
- An exercise on the first homework set showed that

$$f(n_o + kf(n_o)) = 0 \mod f(n_o).$$
• Let \( f \in \mathbb{Z}[X_1, \ldots, X_k] \), a polynomial in \( k \) variables with integer coefficients, be given. Suppose that we have \( k \) pairs of integer values that are congruent modulo some \( n \),
\[(x_1, \ldots, x_k) = (y_1, \ldots, y_k) \mod n, \text{ componentwise.}\]
Then also, because the map \( \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \) is a ring homomorphism,
\[f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k) \mod n.\]

• (Decimal digits) Since \( 10 = 1 (9) \), it follows that for any decimal digits \( a_0, \ldots, a_n \),
\[\sum_{i=0}^{n} a_i 10^i = \sum_{i=0}^{n} a_i \mod 9.\]
This is the grade school result that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. Since \( 10 = -1 (11) \) a similar result holds for divisibility by 11, but with the alternating sum of the digits.

• (A variant of Euclid’s argument) Any odd \( n \) satisfies \( n = 1 (4) \) or \( n = 3 (4) \).
Suppose that there are only finitely primes \( p = 3 (4); \) call them \( p_i \) for \( i = 1, \ldots, k \). (So here \( p_1 = 3 \).) Consider the odd number \( n = 4p_2 \cdots p_k + 3 \) (note that \( p_1 = 3 \) is excluded).
Then \( n \neq 0 (3) \) and \( n = 3 (p_i) \neq 0 (p_i) \) for \( i = 2, \ldots, k \). Thus none of the \( p_i \) divide \( n \), and neither does 2. It follows that \( n \) is a product of primes \( q = 1 (4). \) But any such product is again \( 1 (4) \), contradicting the fact that \( n = 3 (4) \). The conclusion is that there exist infinitely many primes \( p = 3 (4) \).

6. Euler’s Rule and Fermat’s Little Theorem

**Proposition 6.1** (Euler’s Rule). Let \( a, n \) be integers with \( (a, n) = 1 \). Then
\[a^{\phi(n)} = 1 \mod n.\]

**Proof.** For an elementary proof, let \( x_1, \ldots, x_{\phi(n)} \) be the elements of \( \{0, \ldots, n-1\} \) that are coprime to \( n \). Then we have
\[ax_i = x_{j(i)} \mod n, \quad i = 1, \ldots, \phi(n),\]
where the map \( i \mapsto j(i) \) permutes \( \{1, \ldots, \phi(n)\} \). Here one point is that the conditions \( (a, n) = (x_i, n) = 1 \) give \( (ax_i, n) = 1 \) by the observation after Euclid’s Lemma toward the beginning of this writeup; thus we do have \( ax_i = x_{j(i)} \mod n \) for some \( j(i) \). The second point is that if \( ax_i = ax_{i'} (n) \) then \( x_i = x_{i'} (n) \) again by Euclid’s Lemma, because \( a \) is coprime to \( n \), and so \( x_i = x_{i'} \) because both come from \( \{0, \ldots, n-1\} \). Thus \( i = i' \), and the map \( i \mapsto j(i) \) is a permutation, as claimed. Now, because the map is a permutation, we have
\[\prod_{i=1}^{\phi(n)} x_i = \prod_{i=1}^{\phi(n)} (ax_i) = a^{\phi(n)} \prod_{i=1}^{\phi(n)} x_i.\]
Now \( 1 = a^{\phi(n)} (n) \), because the “particular instance of the consequence of Euclid’s Lemma” near the beginning of the writeup says that \( \prod_{i=1}^{\phi(n)} x_i \) is coprime to \( n \). \( \square \)
The points addressed by explicit situational use of Euclid’s Lemma in the previous argument are handled tacitly and automatically by the group structure of \((\mathbb{Z}/n\mathbb{Z})^\times\). Let \(G\) denote this group. Instead of making the elementary argument, let \(x_1, \ldots, x_{\phi(n)}\) be the elements of \(G\), and identify \(a\) with its image \(\pi\) in \(G\). The map \(x \mapsto ax\) permutes \(G\), and so

\[
\prod_{x \in G} ax = \prod_{x \in G} x.
\]

The left side is \(a^{\phi(n)} \prod_{x \in G} x\), and now multiplying by the inverse of the product, or noting that the cancellation law always holds in a group because we can do so in general, gives the result.

Even more generally, Euler’s Rule is a special case consequence of the beginning finite group theory result that the order of every subgroup divides the order of the group. Indeed, for any group element, the order of the cyclic subgroup that it generates divides the order of the group; the order of the cyclic subgroup is the order of its generator, so this latter order divides the order of the group. Thus, raising any group element to the order of the group gives 1. Specializing the group to \((\mathbb{Z}/n\mathbb{Z})^\times\) gives Euler’s rule.

**Corollary 6.2** (Fermat’s Little Theorem). Let \(p\) be prime. For every integer \(a\) such that \(p \nmid a\),

\[
a^{p-1} = 1 \mod p.
\]

In consequence of Fermat’s Little Theorem,

\[
a^p = a \mod p\]

for all integers \(a\) and primes \(p\).

However, the slight gain of information here occurs outside the group setting of \((\mathbb{Z}/p\mathbb{Z})^\times\).

7. The Equation \(ax + ny = b\)

For the equation in this section’s header, we assume that \(a, n, b \in \mathbb{Z}\) and \(n \neq 0\). By now we know that the equation has an integer solution \([x_0, y_0]\) if and only if \((a, n) | b\). When such a solution exists, we obtain the general solution by using linear algebra over \(\mathbb{Q}\). Our equation rewrites as

\[
Av = b, \quad A = \begin{bmatrix} a & n \end{bmatrix}, \quad v = \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Its general solution in \(\mathbb{Q}^2\) is the particular solution \([x_0, y_0]\) plus the solutions of the homogeneous equation, this being

\[
ax + ny = 0.
\]

Unlike the original equation, the homogeneous equation can be divided through by the greatest common divisor \(g\) of \(a\) and \(n\) to get

\[
(a/g)x + (n/g)y = 0, \quad \gcd(a/g, n/g) = 1.
\]

The rational solutions here are \(\mathbb{Q}[n/g, -a/g]\), and the integer solutions \(\mathbb{Z}[n/g, -a/g]\) by the coprimality. Thus the general integer solution of the original equation is

\[
[x \ y] = [x_0 \ y_0] + \mathbb{Z}[n/g, -a/g], \quad g = \gcd(a, n).
\]

Especially, the \(x\)-coordinates of the solutions are

\[
x = x_0 + \mathbb{Z}n/g.
\]
8. The Congruence $ax = b \mod n$

Again consider $a, b, n \in \mathbb{Z}$ with $n \neq 0$. For any $x \in \mathbb{Z}$ we have the equivalences

$$ax = b \mod n \iff ax + ny = b \text{ for some } y.$$ 

So the work that we just did shows the following result.

**Proposition 8.1.** Let $a, b, n \in \mathbb{Z}$ with $n \neq 0$, and let $g = \gcd(a, n)$. The congruence

$$ax = b \mod n$$

has solutions if and only if $g \mid b$. When the congruence has a solution $x_o \in \mathbb{Z}$ then the full solution set is

$$\{x_o + tn/g : t \in \mathbb{Z}\}.$$ 

It follows that the equation $ax = b$ in $\mathbb{Z}/n\mathbb{Z}$ has $g$ solutions. In particular, if $g = 1$ then the equation $ax = b$ has one solution in $\mathbb{Z}/n\mathbb{Z}$.

Perhaps the proposition is most easily remembered as follows:

To solve the congruence

$$ax = b \mod n,$$

let

$$g = \gcd(a, n), \quad a = a'g, \quad n = n'g.$$ 

Then the congruence is

$$a'gx = b \mod n'g.$$ 

Unless $b = b'g$ there are no solutions. If $b = b'g$ then the congruence becomes

$$a'x = b' \mod n', \quad \gcd(a', n') = 1,$$

with unique solution

$$x_o = a'^{-1}b' \mod n'.$$

Thus the original congruence has solutions $x_o + tn' \mod n$, $t = 0, 1, \ldots, g - 1$. 