THE GAUSS SUM NORM VIA FOURIER ANALYSIS

Let p be an odd prime. Consider a primitive Dirichlet character modulo p,

$$\chi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

and let $\zeta_p = e^{2\pi i/p}$. Typically in a first number theory course the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)\zeta_p^a$$

is shown by an elementary argument to satisfy

$$|\tau(\chi)| = \sqrt{p}.$$

This writeup shows that this Gauss sum norm is a small instance of the Plancheral theorem of Fourier analysis.

Extend χ from $(\mathbb{Z}/p\mathbb{Z})^\times$ to $\mathbb{Z}/p\mathbb{Z}$ by defining $\chi(0) = 0$. View $\mathbb{Z}/p\mathbb{Z}$ as an additive group of measure 1 by giving each of its points measure $1/p$. The Fourier transform of χ is

$$\widehat{\chi} : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{C}, \qquad \widehat{\chi}(b) = \frac{1}{\sqrt{p}} \sum_{a=0}^{p-1} \chi(a)\zeta_p^{ba}.$$

So also $\widehat{\chi}(0) = 0$. Fourier inversion in this context is that $\frac{1}{\sqrt{p}}\sum_{b=0}^{p-1} \widehat{\chi}(b)\zeta_p^{-bc} = \chi(c)$ for any $c \in \mathbb{Z}/p\mathbb{Z}$. Indeed,

$$\frac{1}{\sqrt{p}} \sum_{b=0}^{p-1} \widehat{\chi}(b)\zeta_p^{-bc} = \frac{1}{p} \sum_{b,a=0}^{p-1} \chi(a)\zeta_p^{b(a-c)} = \sum_{a=0}^{p-1} \chi(a) \frac{1}{p} \sum_{b=0}^{p-1} \zeta_p^{b(a-c)} = \chi(c).$$

The squared L^2-norm of χ is

$$|\chi|_{L^2}^2 = \frac{1}{p} \sum_{a=1}^{p-1} |\chi(a)|^2 = \frac{p-1}{p}\,.$$

On the other hand, because $\widehat{\chi}(b) = \frac{1}{\sqrt{p}}\sum_{a=1}^{p-1} \chi(a)\zeta_p^{ba} = \frac{\overline{\chi}(b)}{\sqrt{p}}\tau(\chi)$, the squared L^2-norm of $\widehat{\chi}$ is

$$|\widehat{\chi}|_{L^2}^2 = \frac{1}{p} \sum_{b=1}^{p-1} |\widehat{\chi}(b)|^2 = \frac{p-1}{p^2}\,|\tau(\chi)|^2\,.$$

The Plancheral theorem says very generally that the Fourier transform is an isometry, and so $|\chi|_{L^2}^2 = |\widehat{\chi}|_{L^2}^2$, giving $|\tau| = \sqrt{p}$ by the previous two displays.

For the quadratic character, the relations $|\tau|^2 = p$ and $\tau^2 = p^*$ are essentially the same thing, because for a general character χ, noting that $\chi(-1) = \pm 1$ is real,

$$\overline{\tau(\chi)} = \sum_{a=1}^{p-1} \overline{\chi}(a)\zeta_p^{-a} = \overline{\chi}(-1) \sum_{a=1}^{p-1} \overline{\chi}(-a)\zeta_p^{-a} = \chi(-1)\tau(\overline{\chi}),$$

and the quadratic character $(\cdot/p)$ is its own complex conjugate, so $|\tau|^2 = (-1/p)\tau^2$.