FOURIER TRANSFORM OF THE GAUSSIAN

The (one-dimensional) Gaussian function is

$$g : \mathbf{R} \longrightarrow \mathbf{R}, \quad g(x) = e^{-\pi x^2},$$

and it is characterized by the conditions

$$g'(x) = -2\pi x\, g(x), \quad g(0) = 1.$$

The Fourier transform of the Gaussian is

$$\widehat{g} : \mathbf{R} \longrightarrow \mathbf{R}, \quad \widehat{g}(\xi) = \int_{\mathbf{R}} g(x) e^{-2\pi i \xi x}\, dx.$$

(Note that $\widehat{g}$ is real-valued because g is even.) Differentiate under the integral sign to obtain

$$\widehat{g}'(\xi) = \int_{\mathbf{R}} (-2\pi i x) g(x) e^{-2\pi i \xi x}\, dx.$$

Although the factor $-2\pi i x$ comes from the ξ-derivative of $e^{-2\pi i \xi x}$, also $-2\pi x\, g(x)$ is the x-derivative $g'(x)$—this fact is the characterizing differential equation of the Gaussian, as above. Thus, to integrate by parts let

$$dv = -2\pi x\, g(x)\, dx, \qquad u = i e^{-2\pi i \xi x},$$

so that

$$v = g(x), \qquad du = 2\pi \xi e^{-2\pi i \xi x}\, dx,$$

and consequently

$$\widehat{g}'(\xi) = -2\pi \xi \int_{\mathbf{R}} g(x) e^{-2\pi i \xi x}\, dx = -2\pi \xi\, \widehat{g}(\xi).$$

Also, $\widehat{g}(0) = 1$. Thus $\widehat{g}$ satisfies the characterizing properties of the Gaussian. That is, the Gaussian is its own Fourier transform,

$$\widehat{g} = g.$$