FOURIER TRANSFORM OF THE GAUSSIAN

The (one-dimensional) Gaussian function is
\[ g : \mathbb{R} \rightarrow \mathbb{R}, \quad g(x) = e^{-\pi x^2}, \]
and it is characterized by the conditions
\[ g'(x) = -2\pi x g(x), \quad g(0) = 1. \]

For any \( \xi \in \mathbb{R} \) the frequency-\( \xi \) oscillation is
\[ \psi_\xi : \mathbb{R} \rightarrow \mathbb{C}, \quad \psi_\xi(x) = e^{2\pi i \xi x}. \]

The Fourier transform of the Gaussian is
\[ \mathcal{F} g : \mathbb{R} \rightarrow \mathbb{R}, \quad \mathcal{F} g(\xi) = \int_{\mathbb{R}} g(x) \overline{\psi_\xi(x)} \, dx. \]
(Note that \( \mathcal{F} g \) is real-valued because \( g \) is even.) Differentiate under the integral sign to obtain
\[ (\mathcal{F} g)'(\xi) = \int_{\mathbb{R}} (-2\pi i x) g(x) \overline{\psi_\xi(x)} \, dx. \]

Although the factor \(-2\pi i x\) comes from the \( \xi \)-derivative of \( \overline{\psi_\xi(x)} \), also \(-2\pi x g(x)\) is the \( x \)-derivative \( g'(x) \): this is the characterizing differential equation of the Gaussian, as above. Thus we have
\[
(\mathcal{F} g)'(\xi) = i \int_{\mathbb{R}} g'(x) \overline{\psi_\xi(x)} \, dx \\
= -i \int_{\mathbb{R}} g(x) \overline{\psi'_\xi(x)} \, dx \\
= -2\pi \xi \int_{\mathbb{R}} g(x) \overline{\psi_\xi(x)} \, dx \\
= -2\pi \xi \mathcal{F} g(\xi).
\]
Also, \( \mathcal{F} g(0) = 1 \). Thus \( \mathcal{F} g \) satisfies the characterizing properties of the Gaussian. That is, the Gaussian is its own Fourier transform,
\[ \mathcal{F} g = g. \]