

CONTINUATIONS AND FUNCTIONAL EQUATIONS

The Riemann zeta function is *initially* defined as a sum,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re}(s) > 1.$$

The first part of this writeup gives Riemann's argument that the closely related function

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \operatorname{Re}(s) > 1$$

has a meromorphic continuation to the full s -plane, analytic except for simple poles at $s = 0$ and $s = 1$, and the continuation satisfies the functional equation

$$\xi(s) = \xi(1 - s), \quad s \in \mathbf{C}.$$

The continuation is no longer defined by the sum.

Essentially the same ideas apply to Dirichlet L -functions,

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \quad \operatorname{Re}(s) > 1.$$

The second part of this writeup will give their continuation and functional equation.

Part 1. RIEMANN ZETA: MEROMORPHIC CONTINUATION AND FUNCTIONAL EQUATION

1. THE FOURIER TRANSFORM

The space of measurable and absolutely integrable functions on \mathbf{R} is

$$\mathcal{L}^1(\mathbf{R}) = \{\text{measurable } f : \mathbf{R} \rightarrow \mathbf{C} : \int_{x \in \mathbf{R}} |f(x)| dx < \infty\}.$$

Any $f \in \mathcal{L}^1(\mathbf{R})$ has a *Fourier transform* $\mathcal{F}f : \mathbf{R} \rightarrow \mathbf{C}$ given by

$$\mathcal{F}f(x) = \int_{y \in \mathbf{R}} f(y) e^{-2\pi i y x} dy.$$

Although the Fourier transform is continuous, it need not belong to $\mathcal{L}^1(\mathbf{R})$. But if also $\int_{x \in \mathbf{R}} |f(x)|^2 dx < \infty$ then $\int_{x \in \mathbf{R}} |\mathcal{F}f(x)|^2 dx < \infty$. That is, if $f \in \mathcal{L}^1(\mathbf{R}) \cap \mathcal{L}^2(\mathbf{R})$ then $\mathcal{F}f \in \mathcal{L}^2(\mathbf{R})$.

Conceptually the Fourier transform value $\mathcal{F}f(x) \in \mathbf{C}$ is a sort of inner product of f and the frequency- x oscillation $\psi_x(y) = e^{2\pi i y x}$. Thus we might hope to resynthesize f from the continuum of oscillations weighted suitably by the inner products,

$$f(y) = \int_{x \in \mathbf{R}} \mathcal{F}f(x) e^{2\pi i x y} dx, \quad y \in \mathbf{R}.$$

However, the question of which functions f satisfy the previous display, and the analysis of showing that they do, is nontrivial.

2. THE GAUSSIAN; FOURIER TRANSFORM OF THE GAUSSIAN AND ITS DILATIONS

Let $g \in \mathcal{L}^1(\mathbf{R})$ be the *Gaussian function*,

$$g(x) = e^{-\pi x^2}.$$

We need its Fourier transform. Compute that

$$\mathcal{F}g(x) = \int_{y=-\infty}^{\infty} e^{-\pi(y^2+2iyx-x^2)} e^{-\pi x^2} dy = e^{-\pi x^2} \int_{y=-\infty}^{\infty} e^{-\pi(y+ix)^2} dy.$$

That is, the Fourier transform of the Gaussian g is again g but scaled by an integral. The scaling integral is an integral of g extended to the complex plane, taken over a horizontal line translated vertically up from \mathbf{R} . A small exercise in complex contour integration shows that consequently the integral is just the Gaussian integral $\int_{-\infty}^{\infty} e^{-\pi y^2} dy$, which is 1. Thus the Gaussian is its own Fourier transform,

$$\mathcal{F}g = g.$$

Also, for any function $f \in \mathcal{L}^1(\mathbf{R})$ and any positive number r , the r -dilation of f ,

$$f_r(x) = f(xr),$$

has Fourier transform

$$\mathcal{F}(f_r) = r^{-1}(\mathcal{F}f)_{r^{-1}}.$$

So in particular, returning to the Gaussian function g ,

$$\text{the Fourier transform of } g(xt^{1/2}) \text{ is } t^{-1/2}g(xt^{-1/2}), \quad t > 0.$$

3. THE THETA FUNCTION; ITS EXPRESSION AS A SUM OF DILATED GAUSSIANS

Let \mathcal{H} denote the complex upper half plane. The *theta function* on \mathcal{H} is

$$\vartheta : \mathcal{H} \longrightarrow \mathbf{C}, \quad \vartheta(\tau) = \sum_{n \in \mathbf{Z}} e^{\pi i n^2 \tau}.$$

The sum converges very rapidly away from the real axis, making absolute and uniform convergence on compact subsets of \mathcal{H} easy to show, and thus defining a holomorphic function. Specialize to $\tau = it$ with $t > 0$, and write $\theta(t)$ for $\vartheta(it)$. Again let g be the Gaussian. The theta function along the positive imaginary axis is a sum of dilated Gaussians,

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n \in \mathbf{Z}} e^{-\pi n^2 t}, \quad t > 0.$$

This is a sum of quickly decreasing functions whose graphs narrow as n grows.

4. POISSON SUMMATION; THE TRANSFORMATION LAW OF THE THETA FUNCTION

For any function $f \in \mathcal{L}^1(\mathbf{R})$ such that the sum $\sum_{d \in \mathbf{Z}} f(x+d)$ converges absolutely and uniformly on compact sets and is infinitely differentiable as a function of x , the *Poisson summation formula* is

$$\sum_{n \in \mathbf{Z}} f(x+n) = \sum_{n \in \mathbf{Z}} \mathcal{F}f(n) e^{2\pi i n x}.$$

The idea here is that the left side is the periodicization of f , and then the right side is the Fourier series of the left side, because the n th Fourier coefficient of the periodicization of f is the n th Fourier transform of f itself.

More specifically, the \mathbf{Z} -periodicization of f ,

$$F : \mathbf{R} \longrightarrow \mathbf{C}, \quad F(x) = \sum_{n \in \mathbf{Z}} f(x + n),$$

is reproduced by its Fourier series,

$$F(x) = \sum_{n \in \mathbf{Z}} \widehat{F}(n) e^{2\pi i n x}.$$

But as mentioned, the n th Fourier coefficient of F is the n th Fourier transform of f ,

$$\begin{aligned} \widehat{F}(n) &= \int_{t=0}^1 F(t) e^{-2\pi i n t} dt = \int_{t=0}^1 \sum_{k \in \mathbf{Z}} f(t+k) e^{-2\pi i n (t+k)} dt \\ &= \int_{t=-\infty}^{\infty} f(t) e^{-2\pi i n t} dt = \mathcal{F}f(n), \end{aligned}$$

and so the identity $F(x) = \sum_{n \in \mathbf{Z}} \widehat{F}(n) e^{2\pi i n x}$ give the Poisson summation formula as claimed,

$$\sum_{n \in \mathbf{Z}} f(x+n) = \sum_{n \in \mathbf{Z}} \mathcal{F}f(n) e^{2\pi i n x}.$$

When $x = 0$ the Poisson summation formula specializes to

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \mathcal{F}f(n).$$

And especially, if $f(x)$ is the Gaussian $g(xt^{1/2})$ then Poisson summation with $x = 0$ shows that

$$\sum_{n \in \mathbf{Z}} g(nt^{1/2}) = t^{-1/2} \sum_{n \in \mathbf{Z}} g(nt^{-1/2}),$$

which is to say,

$$\boxed{\theta(1/t) = t^{1/2} \theta(t), \quad t > 0.}$$

The previous display says that the theta function is a *modular form*.

As we will see in the second part of this writeup, Poisson summation without specializing to $x = 0$ similarly shows that a more general theta function satisfies a more complicated transformation law.

5. RIEMANN ZETA AS THE MELLIN TRANSFORM OF THETA

With these preliminaries in hand, the properties of the Riemann zeta function are established by examining the *Mellin transform* of (essentially) the theta function. In general, the Mellin transform of a function $f : \mathbf{R}^+ \longrightarrow \mathbf{C}$ is the integral

$$g(s) = \int_{t=0}^{\infty} f(t) t^{s/2} \frac{dt}{t}$$

for s -values such that the integral converges absolutely. (So here g no longer denotes the Gaussian.) The Mellin transform is merely the Fourier transform in different coordinates, as is explained in a different writeup. For example, the Mellin transform of e^{-t} is $\Gamma(s/2)$. Also, the Mellin transform of the function

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{2}(\theta(t) - 1), \quad t > 0$$

is

$$g(s) = \int_{t=0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}.$$

Since $\theta(t) \rightarrow 1$ as $t \rightarrow \infty$, the modular transformation law $\theta(1/t) = t^{1/2} \theta(t)$ shows that $\theta(t) \sim t^{-1/2}$ as $t \rightarrow 0$, $\theta(t)$, and therefore the integral $g(s)$ converges at its left endpoint if $\operatorname{Re}(s) > 1$. And since the convergence of $\theta(t)$ to 1 as $t \rightarrow \infty$ is rapid, the integral converges at its right end for all values of s . Rapid convergence lets the sum pass through the integral in the previous display to yield, after a change of variable,

$$g(s) = \sum_{n=1}^{\infty} (\pi n^2)^{-s/2} \int_{t=0}^{\infty} e^{-t} t^{s/2} \frac{dt}{t} = \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \operatorname{Re}(s) > 1.$$

Thus, when $\operatorname{Re}(s) > 1$, the integral $g(s)$ is the function $\xi(s)$ mentioned at the beginning of this writeup. On the other hand, recall that the function whose Mellin transform we took is essentially the theta function,

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{2}(\theta(t) - 1), \quad t > 0.$$

So this paragraph has in fact shown that the modified zeta function

$$\xi(s) \stackrel{\text{def}}{=} \pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \operatorname{Re}(s) > 1$$

has an integral representation as the Mellin transform of the theta function,

$$\xi(s) = \frac{1}{2} \int_{t=0}^{\infty} (\theta(t) - 1) t^{s/2} \frac{dt}{t}, \quad \operatorname{Re}(s) > 1.$$

Thinking in these terms, the factor $\pi^{-s/2} \Gamma(s/2)$ is intrinsically associated to $\zeta(s)$, making $\xi(s)$ the natural function to consider. Modern adelic considerations make the factor even more natural as a completion of the zeta function at the infinite prime, but those ideas are beyond our current scope.

6. MEROMORPHIC CONTINUATION AND FUNCTIONAL EQUATION

The facts that ξ is essentially the Mellin transform of θ and that θ is a modular form quickly give rise to the meromorphic continuation and functional equation of ξ . Specifically, compute part of the integral representation of ξ by splitting off a term, replacing t by $1/t$, using the modular transformation law $\theta(1/t) = t^{1/2} \theta(t)$, and splitting off another term to resymmetrize,

$$\begin{aligned} \frac{1}{2} \int_{t=0}^1 (\theta(t) - 1) t^{s/2} \frac{dt}{t} &= \frac{1}{2} \int_{t=0}^1 \theta(t) t^{s/2} \frac{dt}{t} - \frac{1}{s} \\ &= \frac{1}{2} \int_{t=1}^{\infty} \theta(1/t) t^{-s/2} \frac{dt}{t} - \frac{1}{s} \\ &= \frac{1}{2} \int_{t=1}^{\infty} \theta(t) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} \\ &= \frac{1}{2} \int_{t=1}^{\infty} (\theta(t) - 1) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}. \end{aligned}$$

Combine this with the remainder of the integral representation of $\xi(s)$ to get

$$\xi(s) = \frac{1}{2} \int_{t=1}^{\infty} (\theta(t) - 1)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad \operatorname{Re}(s) > 1.$$

And now, since the integral in the last display now has as its left endpoint of integration $t = 1$ rather than $t = 0$, it is entire in s , making the right side meromorphic everywhere in the s -plane with its only poles being simple poles at $s = 0$ and $s = 1$. That is, the new description of ξ is no longer constrained to the domain $\{\operatorname{Re}(s) > 1\}$,

$$\boxed{\xi(s) = \frac{1}{2} \int_{t=1}^{\infty} (\theta(t) - 1)(t^{s/2} + t^{(1-s)/2}) \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad s \in \mathbf{C}.}$$

The new description extends ξ to a meromorphic function on all of \mathbf{C} . The definition of the extended function no longer makes reference to $\zeta(s)$ as a sum.

Finally, the right side of the boxed display is clearly invariant under the substitution $s \mapsto 1-s$. That is, the meromorphic continuation of $\xi(s)$ to the full s -plane satisfies the functional equation

$$\boxed{\xi(1-s) = \xi(s), \quad s \in \mathbf{C}.}$$

Part 2. DIRICHLET L -FUNCTIONS: ANALYTIC CONTINUATION AND FUNCTIONAL EQUATION

7. THE THETA FUNCTION OF A PRIMITIVE DIRICHLET CHARACTER

A Dirichlet character

$$\chi : (\mathbf{Z}/N\mathbf{Z})^\times \longrightarrow \mathbf{C}^\times$$

is called *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

A primitive even Dirichlet character modulo N has an associated theta function

$$\boxed{\theta_+(t, \chi) \stackrel{\text{def}}{=} \sum_{n \in \mathbf{Z}} \chi(n) e^{-\pi n^2 t / N}, \quad t > 0.}$$

The sum $\theta_+(t, \chi)$ is zero for odd χ . A primitive odd Dirichlet character modulo N has an associated theta function

$$\boxed{\theta_-(t, \chi) \stackrel{\text{def}}{=} \sum_{n \in \mathbf{Z}} n \chi(n) e^{-\pi n^2 t / N}, \quad t > 0.}$$

The sum $\theta_-(t, \chi)$ is zero for even χ .

We will derive modular transformation laws for these theta functions.

8. TWO POISSON SUMMATION RESULTS

Recall that the Poisson summation formula says that for suitable functions $f : \mathbf{R} \longrightarrow \mathbf{C}$ (in particular, for Schwartz functions)

$$\sum_{n \in \mathbf{Z}} f(x+n) = \sum_{n \in \mathbf{Z}} \mathcal{F}f(n) e^{2\pi i n x}, \quad x \in \mathbf{R}.$$

Recall also that the Fourier transform of a dilation $f_r(x) = f(rx)$ of a suitable function f is

$$\mathcal{F}(f_r) = (1/r)(\mathcal{F}f)_{1/r}, \quad r > 0.$$

And recall that the Gaussian function,

$$g : \mathbf{R} \longrightarrow \mathbf{R}, \quad g(x) = e^{-\pi x^2},$$

is its own Fourier transform, i.e., $\mathcal{F}g = g$.

Using the results just mentioned, compute that for $x \in \mathbf{R}$ and $r > 0$,

$$\begin{aligned} \sum_{n \in \mathbf{Z}} e^{-\pi(x+n)^2/r} &= \sum_{n \in \mathbf{Z}} g_{r^{-1/2}}(x+n) \\ &= \sum_{n \in \mathbf{Z}} (\mathcal{F}g_{r^{-1/2}})(n) e^{2\pi i n x} && \text{by Poisson summation} \\ &= r^{1/2} \sum_{n \in \mathbf{Z}} (\mathcal{F}g)_{r^{1/2}}(n) e^{2\pi i n x} && \text{by the dilation formula} \\ &= r^{1/2} \sum_{n \in \mathbf{Z}} g_{r^{1/2}}(n) e^{2\pi i n x} && \text{by the property of the Gaussian} \\ &= r^{1/2} \sum_{n \in \mathbf{Z}} e^{2\pi i n x - \pi n^2 r} \end{aligned}$$

A slight rearrangement gives

$$\sum_{n \in \mathbf{Z}} e^{2\pi i n x - \pi n^2 r} = r^{-1/2} \sum_{n \in \mathbf{Z}} e^{-\pi(x+n)^2/r}, \quad x \in \mathbf{R}, \quad r > 0.$$

Differentiate with respect to x to get

$$\sum_{n \in \mathbf{Z}} n e^{2\pi i n x - \pi n^2 r} = i r^{-3/2} \sum_{n \in \mathbf{Z}} (x+n) e^{-\pi(x+n)^2/r}, \quad x \in \mathbf{R}, \quad r > 0.$$

Although the previous two displays bear some resemblance to modular transformation laws for the theta functions of even and odd primitive Dirichlet characters, to make things dovetail perfectly we also need to consider Gauss sums.

9. THE GAUSS SUMS OF PRIMITIVE DIRICHLET CHARACTERS

A primitive Dirichlet character

$$\chi : (\mathbf{Z}/N\mathbf{Z})^\times \longrightarrow \mathbf{C}^\times$$

has associated Gauss sums

$$\tau_n(\chi) = \sum_{m=0}^{N-1} \chi(m) e^{2\pi i n m / N}, \quad n \in \mathbf{Z}.$$

Note that $\tau_n(\chi) = \mathcal{F}\chi(n)$, viewing the basic character of $\mathbf{Z}/N\mathbf{Z}$ as $\psi(x) = e^{-2\pi i x / N}$. Especially, the basic Gauss sum associated to χ is

$$\tau(\chi) = \tau_1(\chi) = \mathcal{F}\chi(1) = \sum_{m=0}^{N-1} \chi(m) e^{2\pi i m / N}.$$

A standard identity is

$$\tau_n(\chi) = \bar{\chi}(n) \tau(\chi), \quad n \in \mathbf{Z}.$$

For n coprime to N the proof is immediate, but when $(n, N) > 1$ the primitivity of χ is necessary for the argument. A consequence of the previous display is

$$(1) \quad \tau(\chi) \tau(\bar{\chi}) = \chi(-1) N.$$

Recall that a Dirichlet character χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$. For a primitive even Dirichlet character, (1) becomes

$$\tau(\bar{\chi})/N^{1/2} = N^{1/2}/\tau(\chi),$$

and $\tau(\chi)/N^{1/2} = 1$ if χ is quadratic. For a primitive odd Dirichlet character, (1) becomes

$$\tau(\bar{\chi})/(iN^{1/2}) = iN^{1/2}/\tau(\chi),$$

and $\tau(\chi)/(iN^{1/2}) = 1$ if χ is quadratic.

10. TRANSFORMATION LAWS OF DIRICHLET THETA FUNCTIONS

Recall that the theta function of a primitive even Dirichlet character χ modulo N is

$$\theta_+(t, \chi) \stackrel{\text{def}}{=} \sum_{n \in \mathbf{Z}} \chi(n) e^{-\pi n^2 t/N}, \quad t > 0.$$

Compute (using the identity $\tau_n(\chi) = \bar{\chi}(n)\tau(\chi)$ but with $\bar{\chi}$ in place of χ for the second equality)

$$\begin{aligned} \tau(\bar{\chi})\theta_+(t, \chi) &= \sum_{n \in \mathbf{Z}} \chi(n)\tau(\bar{\chi})e^{-\pi n^2 t/N} = \sum_{n \in \mathbf{Z}} \tau_n(\bar{\chi})e^{-\pi n^2 t/N} \\ &= \sum_{n \in \mathbf{Z}} \sum_{m=0}^{N-1} \bar{\chi}(m) e^{2\pi i n m/N} e^{-\pi n^2 t/N} \\ &= \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} e^{2\pi i n m/N - \pi n^2 t/N}. \end{aligned}$$

Apply the relation $\sum_{n \in \mathbf{Z}} e^{2\pi i n x - \pi n^2 r} = r^{-1/2} \sum_{n \in \mathbf{Z}} e^{-\pi(x+n)^2/r}$ from Poisson summation, with $x = m/N$ and $r = t/N$,

$$\begin{aligned} \tau(\bar{\chi})\theta_+(t, \chi) &= (N/t)^{1/2} \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} e^{-\pi(m/N+n)^2 N/t} \\ &= (N/t)^{1/2} \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} e^{-\pi(m+nN)^2 (1/t)/N} \\ &= (N/t)^{1/2} \sum_{\ell \in \mathbf{Z}} \bar{\chi}(\ell) e^{-\pi \ell^2 (1/t)/N} \\ &= (N/t)^{1/2} \theta_+(1/t, \bar{\chi}). \end{aligned}$$

A slight rearrangement, replacing χ by $\bar{\chi}$, gives the modular transformation law in the even case,

$$\boxed{\theta_+(1/t, \chi) = \tau(\chi)/N^{1/2} \cdot t^{1/2} \theta_+(t, \bar{\chi}), \quad t > 0.}$$

Recall that the theta function of a primitive odd Dirichlet character χ modulo N is

$$\theta_-(t, \chi) \stackrel{\text{def}}{=} \sum_{n \in \mathbf{Z}} n\chi(n) e^{-\pi n^2 t/N}, \quad t > 0.$$

Compute similarly to the even case that

$$\tau(\bar{\chi})\theta_-(t, \chi) = \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} n e^{2\pi i n m / N - \pi n^2 t / N},$$

and then apply the relation $\sum_{n \in \mathbf{Z}} n e^{2\pi i n x - \pi n^2 r} = i r^{-3/2} \sum_{n \in \mathbf{Z}} (x+n) e^{-\pi(x+n)^2 / r}$ from Poisson summation, with $x = m/N$ and $r = t/N$,

$$\begin{aligned} \tau(\bar{\chi})\theta_-(t, \chi) &= i(N/t)^{3/2} \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} (m/N + n) e^{-\pi(m/N + n)^2 N/t} \\ &= iN^{1/2} t^{-3/2} \sum_{m=0}^{N-1} \bar{\chi}(m) \sum_{n \in \mathbf{Z}} (m + nN) e^{-\pi(m+nN)^2 (1/t)/N} \\ &= iN^{1/2} t^{-3/2} \sum_{\ell \in \mathbf{Z}} \ell \bar{\chi}(\ell) e^{-\pi \ell^2 (1/t)/N} \\ &= iN^{1/2} t^{-3/2} \theta_-(1/t, \bar{\chi}). \end{aligned}$$

Again rearrange, replacing χ by $\bar{\chi}$, to get the modular transformation law in the odd case,

$$\boxed{\theta_-(1/t, \chi) = \tau(\chi) / (iN^{1/2}) \cdot t^{3/2} \theta_-(t, \bar{\chi}), \quad t > 0.}$$

11. THE FUNCTIONAL EQUATION FOR χ EVEN

Let χ be a primitive even Dirichlet character modulo $N > 1$, meaning that $\chi(-1) = 1$ and $\chi(0) = 0$. For $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 0$ and for $n \in \mathbf{Z}^+$ we have the identity

$$\Gamma(s/2) = \int_{t=0}^{\infty} e^{-t} t^{s/2} \frac{dt}{t} = N^{-s/2} \pi^{s/2} n^s \int_{t=0}^{\infty} e^{-\pi n^2 t / N} t^{s/2} \frac{dt}{t},$$

and thus

$$N^{s/2} \pi^{-s/2} \Gamma(s/2) \chi(n) n^{-s} = \int_{t=0}^{\infty} \chi(n) e^{-\pi n^2 t / N} t^{s/2} \frac{dt}{t}.$$

Sum over n and recall that $\chi(-1) = 1$ to get

$$N^{s/2} \pi^{-s/2} \Gamma(s/2) L(s, \chi) = \frac{1}{2} \int_{t=0}^{\infty} \theta_+(t, \chi) t^{s/2} \frac{dt}{t}.$$

The integral converges at $t = 0$ because $\operatorname{Re}(s) > 0$, but it converges at $t = \infty$ independently of the value of s . Compute, using the even modular transformation law $\theta_+(1/t, \chi) = \tau(\chi) / N^{1/2} \cdot t^{1/2} \theta_+(t, \bar{\chi})$ for the third equality, that the integral is

$$\begin{aligned} \int_{t=0}^{\infty} \theta_+(t, \chi) t^{s/2} \frac{dt}{t} &= \int_{t=1}^{\infty} \theta_+(t, \chi) t^{s/2} \frac{dt}{t} + \int_{t=0}^1 \theta_+(t, \chi) t^{s/2} \frac{dt}{t} \\ &= \int_{t=1}^{\infty} (\theta_+(t, \chi) t^{s/2} + \theta_+(1/t, \chi) t^{-s/2}) \frac{dt}{t} \\ &= \int_{t=1}^{\infty} (\theta_+(t, \chi) t^{s/2} + \tau(\chi) / N^{1/2} \theta_+(t, \bar{\chi}) t^{(1-s)/2}) \frac{dt}{t}. \end{aligned}$$

The last integral is an entire function of s . Thus the function

$$\boxed{\xi_+(s, \chi) \stackrel{\text{def}}{=} N^{s/2} \pi^{-s/2} \Gamma(s/2) L(s, \chi), \quad \operatorname{Re}(s) > 1,}$$

which is initially a Mellin transform,

$$\xi_+(s, \chi) = \frac{1}{2} \int_{t=0}^{\infty} \theta_+(t, \chi) t^{s/2} \frac{dt}{t} \quad \operatorname{Re}(s) > 1,$$

extends to an entire function of s , also defined as an integral,

$$\xi_+(s, \chi) = \frac{1}{2} \int_{t=1}^{\infty} (\theta_+(t, \chi) t^{s/2} + \tau(\chi)/N^{1/2} \theta_+(t, \bar{\chi}) t^{(1-s)/2}) \frac{dt}{t}, \quad s \in \mathbf{C}.$$

Since $N^{s/2} \pi^{-s/2} \Gamma(s/2)$ has no zeros, $L(s, \chi)$ extends to an entire function of s as well. Furthermore, because $\tau(\bar{\chi})/N^{1/2} = N^{1/2}/\tau(\chi)$, replacing s by $1-s$ and χ by $\bar{\chi}$ in the last integral multiplies it by $N^{1/2}/\tau(\chi)$,

$$\begin{aligned} & \int_{t=1}^{\infty} (\theta_+(t, \bar{\chi}) t^{(1-s)/2} + \tau(\bar{\chi})/N^{1/2} \theta_+(t, \chi) t^{s/2}) \frac{dt}{t} \\ &= N^{1/2}/\tau(\chi) \int_{t=1}^{\infty} (\theta_+(t, \chi) t^{s/2} + \tau(\chi)/N^{1/2} \theta_+(t, \bar{\chi}) t^{(1-s)/2}) \frac{dt}{t}. \end{aligned}$$

Therefore we have the functional equation

$$(\tau(\chi)/N^{1/2}) \xi_+(1-s, \bar{\chi}) = \xi_+(s, \chi), \quad s \in \mathbf{C}.$$

The functional equation shows that the only zeros of the extended $L(s, \chi)$ in the left half plane are at $s = -2, -4, -6, \dots$, and also that the extended $L(s, \chi)$ has a zero at $s = 0$.

12. THE FUNCTIONAL EQUATION FOR χ ODD

Let χ be a primitive odd Dirichlet character modulo $N > 1$, meaning that $\chi(-1) = -1$ and $\chi(0) = 0$. For $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 0$ and for $n \in \mathbf{Z}^+$ we have the identity

$$\begin{aligned} \Gamma((s+1)/2) &= \int_{t=0}^{\infty} e^{-t} t^{(s+1)/2} \frac{dt}{t} \\ &= N^{-(s+1)/2} \pi^{(s+1)/2} n^{s+1} \int_{t=0}^{\infty} e^{-\pi n^2 t/N} t^{(s+1)/2} \frac{dt}{t}, \end{aligned}$$

and thus

$$N^{(s+1)/2} \pi^{-(s+1)/2} \Gamma((s+1)/2) \chi(n) n^{-s} = \int_{t=0}^{\infty} n \chi(n) e^{-\pi n^2 t/N} t^{(s+1)/2} \frac{dt}{t}.$$

Sum over n and recall that $\chi(-1) = -1$ to get

$$N^{(s+1)/2} \pi^{-(s+1)/2} \Gamma((s+1)/2) L(s, \chi) = \frac{1}{2} \int_{t=0}^{\infty} \theta_-(t, \chi) t^{(s+1)/2} \frac{dt}{t}.$$

As before, the integral converges robustly at $t = \infty$. Compute, using the odd modular transformation law $\theta_-(1/t, \chi) = \tau(\chi)/(iN^{1/2}) \cdot t^{3/2} \theta_-(t, \bar{\chi})$ for the third

equality, that the integral is

$$\begin{aligned}
\int_{t=0}^{\infty} \theta_{-}(t, \chi) t^{(s+1)/2} \frac{dt}{t} &= \int_{t=1}^{\infty} \theta_{-}(t, \chi) t^{(s+1)/2} \frac{dt}{t} + \int_{t=0}^1 \theta_{-}(t, \chi) t^{(s+1)/2} \frac{dt}{t} \\
&= \int_{t=1}^{\infty} (\theta_{-}(t, \chi) t^{(s+1)/2} + \theta_{-}(1/t, \chi) t^{-(s+1)/2}) \frac{dt}{t} \\
&= \int_{t=1}^{\infty} (\theta_{-}(t, \chi) t^{(s+1)/2} + \tau(\chi)/(iN^{1/2}) \theta_{-}(t, \bar{\chi}) t^{(3-s-1)/2}) \frac{dt}{t} \\
&= \int_{t=1}^{\infty} (\theta_{-}(t, \chi) t^{(s+1)/2} + \tau(\chi)/(iN^{1/2}) \theta_{-}(t, \bar{\chi}) t^{(1-s+1)/2}) \frac{dt}{t}.
\end{aligned}$$

The last integral is an entire function of s . Thus the function

$$\xi_{-}(s, \chi) \stackrel{\text{def}}{=} N^{(s+1)/2} \pi^{-(s+1)/2} \Gamma((s+1)/2) L(s, \chi), \quad \text{Re}(s) > 1,$$

which is initially a Mellin transform,

$$\xi_{-}(s, \chi) = \frac{1}{2} \int_{t=0}^{\infty} \theta_{-}(t, \chi) t^{(s+1)/2} \frac{dt}{t}, \quad \text{Re}(s) > 1,$$

extends to an entire function of s , also defined as an integral,

$$\xi_{-}(s, \chi) = \frac{1}{2} \int_{t=1}^{\infty} (\theta_{-}(t, \chi) t^{(s+1)/2} + \tau(\chi)/(iN^{1/2}) \theta_{-}(t, \bar{\chi}) t^{(1-s+1)/2}) \frac{dt}{t}, \quad s \in \mathbf{C}.$$

Consequently $L(s, \chi)$ extends to an entire function of s as well. Furthermore, because $\tau(\bar{\chi})/(iN^{1/2}) = iN^{1/2}/\tau(\chi)$, replacing s by $1-s$ and χ by $\bar{\chi}$ in the last integral multiplies it by $iN^{1/2}/\tau(\chi)$,

$$\begin{aligned}
&\int_{t=1}^{\infty} (\theta_{-}(t, \bar{\chi}) t^{(1-s+1)/2} + \tau(\bar{\chi})/(iN^{1/2}) \theta_{-}(t, \chi) t^{(s+1)/2}) \frac{dt}{t} \\
&= iN^{1/2}/\tau(\chi) \int_{t=1}^{\infty} (\theta_{-}(t, \chi) t^{(s+1)/2} + \tau(\chi)/(iN^{1/2}) \theta_{-}(t, \bar{\chi}) t^{(1-s+1)/2}) \frac{dt}{t}.
\end{aligned}$$

Therefore we have the functional equation

$$(\tau(\chi)/(iN^{1/2})) \xi_{-}(1-s, \bar{\chi}) = \xi_{-}(s, \chi), \quad s \in \mathbf{C}.$$

The functional equation shows that the only zeros of the extended $L(s, \chi)$ are at $s = -1, -3, -5, \dots$.