CONTINUATIONS AND FUNCTIONAL EQUATIONS

The Riemann zeta function is *initially* defined as a sum,

$$\zeta(s) = \sum_{n \geq 1} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

The first part of this writeup gives Riemann's argument that the *completion* of zeta,

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1$$

has a meromorphic continuation to the full s-plane, analytic except for simple poles at $s = 0$ and $s = 1$, and the continuation satisfies the functional equation

$$Z(s) = Z(1 - s), \quad s \in \mathbb{C}.$$

The continuation is no longer defined by the sum. Instead, it is defined by a well-behaved integral-with-parameter.

Essentially the same ideas apply to Dirichlet L-functions,

$$L(\chi, s) = \sum_{n \geq 1} \chi(n) n^{-s}, \quad \mathrm{Re}(s) > 1.$$

The second part of this writeup will give their completion, continuation and functional equation.

Contents

Part 1. RIEMANN ZETA: MEROMORPHIC CONTINUATION AND FUNCTIONAL EQUATION

1. Fourier transform

The space of measurable and absolutely integrable functions on $\mathbb{R}$ is

$$\mathcal{L}^1(\mathbb{R}) = \{\text{measurable } f : \mathbb{R} \longrightarrow \mathbb{C} \ : \ \int_{x\in\mathbb{R}} |f(x)|\,\mathrm{d}x < \infty\}.$$

Any $f \in \mathcal{L}^1(\mathbb{R})$ has a *Fourier transform* $\mathcal{F}f : \mathbb{R} \longrightarrow \mathbb{C}$ given by

$$\mathcal{F}f(\xi) = \int_{x\in\mathbb{R}} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x.$$

Although the Fourier transform is continuous, it needn't belong to $\mathcal{L}^1(\mathbb{R})$. But if also $f \in \mathcal{L}^2(\mathbb{R})$, *i.e.*, $\int_{x\in\mathbb{R}} |f(x)|^2\,\mathrm{d}x < \infty$, then $\int_{x\in\mathbb{R}} |\mathcal{F}f(x)|^2\,\mathrm{d}x < \infty$. That is, if $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$ then $\mathcal{F}f \in \mathcal{L}^2(\mathbb{R})$.

Conceptually the Fourier transform value $\mathcal{F}f(x) \in \mathbb{C}$ is a sort of inner product of f and the frequency-ξ oscillation $\psi_\xi(x) = e^{2\pi i\xi x}$. Thus we might hope to resynthesize f from the continuum of oscillations weighted suitably by the inner products,

$$f(x) = \int_{\xi\in\mathbb{R}} \mathcal{F}f(\xi)e^{2\pi i\xi x}\,\mathrm{d}\xi, \quad x \in \mathbb{R}.$$

However, the question of which functions f satisfy the previous display, and the analysis of showing that they do, are nontrivial.

2. Fourier transform of the Gaussian and its dilations

Let $g \in \mathcal{L}^1(\mathbb{R})$ be the *Gaussian function*,

$$g(x) = e^{-\pi x^2}.$$

The Fourier transform of the Gaussian is again the Gaussian,

$$\mathcal{F}g = g.$$

This is readily shown by complex contour integration or by differentiation under the integral sign, as follows.

For the contour integration argument, compute that

$$\begin{aligned}\mathcal{F}g(\eta) &= \int_{x=-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i\eta x}\,\mathrm{d}x \\ &= \int_{x=-\infty}^{\infty} e^{-\pi(x^2+2ix\eta-\eta^2)}e^{-\pi\eta^2}\,\mathrm{d}x \\ &= e^{-\pi\eta^2}\int_{x=-\infty}^{\infty} e^{-\pi(x+i\eta)^2}\,\mathrm{d}x.\end{aligned}$$

That is, $\mathcal{F}g(\eta)$ is $g(\eta)$ scaled by an integral. The scaling integral is an integral of the extension of g to the complex plane, taken over a horizontal line translated vertically from $\mathbb{R}$. A small exercise with Cauchy's Theorem and limits shows that consequently the integral is just the Gaussian integral $\int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x$, which is 1. Thus $\mathcal{F}g = g$ as claimed.

For the differentiation argument, note that $g'(x) = -2\pi x\,g(x)$ and $g(0) = 1$. Let $\psi_\xi(x) = e^{2\pi i\xi x}$ so that the Fourier transform of the Gaussian is $\mathcal{F}g(\xi) =$

$\int_{x=-\infty}^{\infty} g(x)\overline{\psi}_\xi(x)\,\mathrm{d}x$, and compute, differentiating under the integral sign, recognizing $-2\pi x g(x)$ as $g'(x)$, and integrating by parts,

$$\begin{aligned}(\mathcal{F}g)'(\xi) &= \int_{x=-\infty}^{\infty} g(x)\frac{\partial}{\partial\xi}\overline{\psi}_\xi(x)\,\mathrm{d}x = \int_{x=-\infty}^{\infty}(-2\pi i x)g(x)\overline{\psi}_\xi(x)\,\mathrm{d}x \\ &= i\int_{x=-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}g(x)\overline{\psi}_\xi(x)\,\mathrm{d}x = -i\int_{x=-\infty}^{\infty} g(x)\frac{\partial}{\partial x}\overline{\psi}_\xi(x)\,\mathrm{d}x \\ &= -2\pi\xi\int_{x=-\infty}^{\infty} g(x)\overline{\psi}_\xi(x)\,\mathrm{d}x = -2\pi\xi\,\mathcal{F}g(\xi).\end{aligned}$$

Also $\mathcal{F}g(0) = \int_{x=-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$. Thus $\mathcal{F}g$ satisfies the same differential equation and initial condition as g, and again we have $\mathcal{F}g = g$ as claimed.

For any function $f \in \mathcal{L}^1(\mathbb{R})$ and any positive number r, the r-dilation of f,

$$f_r(x) = f(xr),$$

has Fourier transform

$$\mathcal{F}(f_r) = r^{-1}(\mathcal{F}f)_{r^{-1}}.$$

So in particular, returning to the Gaussian function g,

$$\textit{the Fourier transform of } g_{t^{-1/2}} \textit{ is } t^{1/2}g_{t^{1/2}}, \quad t > 0.$$

3. Theta function

Let $\mathcal{H}$ denote the complex upper half plane. The *theta function* on $\mathcal{H}$ is

$$\vartheta : \mathcal{H} \longrightarrow \mathbb{C}, \qquad \vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

The sum converges very rapidly away from the real axis, making absolute and uniform convergence on compact subsets of $\mathcal{H}$ easy to show, and thus defining a holomorphic function. Specialize to $\tau = it$ with $t > 0$, and write $\theta(t)$ for $\vartheta(it)$. Again let g be the Gaussian. The theta function along the positive imaginary axis is a sum of dilated Gaussians whose graphs narrow as n grows absolutely,

$$\boxed{\theta(t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \quad t > 0.}$$

Equivalently,

$$\boxed{\theta(t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} g_{t^{1/2}}(n), \quad t > 0.}$$

4. Poisson summation; the transformation law of the theta function

For any function $f \in \mathcal{L}^1(\mathbb{R})$ such that the sum $\sum_{d\in\mathbb{Z}} f(x+d)$ converges absolutely and uniformly on compact sets and is infinitely differentiable as a function of x, the *Poisson summation formula* is

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n)e^{2\pi i n x}.$$

The idea here is that the left side is the periodicization of f, and then the right side is the Fourier series of the left side, because the nth Fourier coefficient of the periodicized f is the nth Fourier transform of f itself.

More specifically, the $\mathbb{Z}$-periodicization of f,

$$F : \mathbb{R} \longrightarrow \mathbb{C}, \quad F(x) = \sum_{n\in\mathbb{Z}} f(x+n),$$

is reproduced by its Fourier series,

$$F(x) = \sum_{n\in\mathbb{Z}} \widehat{F}(n) e^{2\pi i n x}.$$

But as mentioned, the nth Fourier coefficient of F is the nth Fourier transform of f,

$$\begin{aligned}\widehat{F}(n) = \int_{t=0}^{1} F(t)e^{-2\pi i n t}\,\mathrm{d}t &= \int_{t=0}^{1} \sum_{k\in\mathbb{Z}} f(t+k)e^{-2\pi i n(t+k)}\,\mathrm{d}t \\ &= \int_{t=-\infty}^{\infty} f(t)e^{-2\pi i n t}\,\mathrm{d}t = \mathcal{F}f(n),\end{aligned}$$

and so the identity $F(x) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi i n x}$ gives the Poisson summation formula as claimed,

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n) e^{2\pi i n x}.$$

When $x = 0$ the Poisson summation formula specializes to

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n).$$

And especially, if f is the dilated Gaussian $g_{t^{-1/2}}$ then Poisson summation with $x = 0$ shows that

$$\sum_{n\in\mathbb{Z}} g_{t^{-1/2}}(n) = t^{1/2} \sum_{n\in\mathbb{Z}} g_{t^{1/2}}(n),$$

which is to say,

$$\boxed{\theta(1/t) = t^{1/2}\,\theta(t), \quad t > 0.}$$

The previous display says that the theta function is a *modular form*.

As we will see in the second part of this writeup, Poisson summation without specializing to $x = 0$ similarly shows that a more general theta function satisfies a more complicated transformation law.

5. Riemann zeta as the Mellin transform of theta

With these preliminaries in hand, the properties of the Riemann zeta function are established by examining the *Mellin transform* of (essentially) the theta function. In general, the Mellin transform of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{C}$ is the integral

$$g(s) = \int_{t=0}^{\infty} f(t) t^s \,\frac{\mathrm{d}t}{t}$$

for s-values such that the integral converges absolutely. Here g no longer denotes the Gaussian. The Mellin transform is merely the Fourier transform in different coordinates, as is explained in another writeup. For example, the Mellin transform of e^{-t} is $\Gamma(s)$. Also, the Mellin transform at $s/2$ of the function

$$\tfrac{1}{2}(\theta(t) - 1) = \sum_{n\geq 1} e^{-\pi n^2 t}, \quad t > 0$$

is

$$g(s/2) = \frac{1}{2}\int_{t=0}^{\infty}(\theta(t)-1)t^{s/2}\,\frac{\mathrm{d}t}{t}\,.$$

Because $\theta(t) \to 1$ as $t \to \infty$, the modular transformation law $\theta(1/t) = t^{1/2}\,\theta(t)$ shows that $\theta(t) \sim t^{-1/2}$ as $t \to 0^+$, making the integrand roughly $t^{(s-1)/2}\,\mathrm{d}t/t$ as $t \to 0^+$, and therefore the integral converges at its left end for $\mathrm{Re}(s) > 1$. Replace $\frac{1}{2}(\theta(t)-1)$ by its expression as a sum to get

$$g(s/2) = \int_{t=0}^{\infty}\sum_{n\geq 1} e^{-\pi n^2 t}t^{s/2}\,\frac{\mathrm{d}t}{t}\,.$$

Because the convergence of $\theta(t)$ to 1 as $t \to \infty$ is rapid, the integral converges at its right end for all values of s. Also, the rapid convergence lets the sum pass through the integral in the previous display to yield, after a change of variable,

$$g(s/2) = \sum_{n\geq 1}(\pi n^2)^{-s/2}\int_{t=0}^{\infty} e^{-t}t^{s/2}\,\frac{\mathrm{d}t}{t} = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1.$$

Thus, when $\mathrm{Re}(s) > 1$, the integral $g(s/2)$ is the function $Z(s)$ mentioned at the beginning of this writeup. So this paragraph has in fact shown that the modified zeta function

$$\boxed{Z(s) \stackrel{\text{def}}{=} \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1}$$

has an integral representation as the Mellin transform of (essentially) the theta function,

$$\boxed{Z(s) = \frac{1}{2}\int_{t=0}^{\infty}(\theta(t)-1)t^{s/2}\,\frac{\mathrm{d}t}{t}\,, \quad \mathrm{Re}(s) > 1.}$$

Thinking in these terms, the factor $\pi^{-s/2}\Gamma(s/2)$ is intrinsically associated to $\zeta(s)$, making $Z(s)$ the natural function to consider. Modern adelic considerations make the factor even more natural as a completion of the zeta function at the infinite prime, but those ideas are beyond our current scope.

6. Meromorphic continuation and functional equation

The facts that Z is essentially the Mellin transform of θ and that θ is a modular form quickly give rise to the meromorphic continuation and functional equation of Z. Specifically, compute part of the integral representation of Z by replacing t by $1/t$ and then using the modular transformation law $\theta(1/t) = t^{1/2}\,\theta(t)$ and the condition $\mathrm{Re}(s) > 1$, and also using the little identity $\int_1^\infty t^\alpha\,\mathrm{d}t/t = -1/\alpha$ for $\mathrm{Re}(\alpha) < 0$,

$$\begin{aligned}\frac{1}{2}\int_{t=0}^{1}(\theta(t)-1)t^{s/2}\,\frac{\mathrm{d}t}{t} &= \frac{1}{2}\int_{t=1}^{\infty}(\theta(1/t)-1)t^{-s/2}\,\frac{\mathrm{d}t}{t}\\ &= \frac{1}{2}\int_{t=1}^{\infty}\Big((\theta(t)-1)t^{(1-s)/2} - t^{-s/2} + t^{(1-s)/2}\Big)\,\frac{\mathrm{d}t}{t}\\ &= \frac{1}{2}\int_{t=1}^{\infty}(\theta(t)-1)t^{(1-s)/2}\,\frac{\mathrm{d}t}{t} - \frac{1}{s} - \frac{1}{1-s}\,.\end{aligned}$$

Combine this with the remainder of the integral representation of $Z(s)$ to get

$$Z(s) = \frac{1}{2}\int_{t=1}^{\infty}(\theta(t)-1)(t^{s/2}+t^{(1-s)/2})\,\frac{\mathrm{d}t}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad \mathrm{Re}(s) > 1.$$

And now, because the integral in the last display has left end $t = 1$ rather than $t = 0$, it is entire in s, making the right side meromorphic everywhere in the s-plane with its only poles being simple poles at $s = 0$ and $s = 1$. That is, the new description of Z is no longer constrained to the right half plane $\mathrm{Re}(s) > 1$,

$$\boxed{Z(s) = \frac{1}{2}\int_{t=1}^{\infty}(\theta(t)-1)(t^{s/2}+t^{(1-s)/2})\,\frac{\mathrm{d}t}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad s \in \mathbb{C}.}$$

This new description extends Z to a meromorphic function on all of $\mathbb{C}$. The definition of the extended function no longer makes reference to $\zeta(s)$ as a sum.

The right side of the previous display is clearly invariant under the substitution $s \mapsto 1-s$. That is, the meromorphic continuation of $Z(s)$ to the full s-plane satisfies the functional equation

$$\boxed{Z(1-s) = Z(s), \quad s \in \mathbb{C}.}$$

The Euler product of $\zeta(s)$ for $\mathrm{Re}(s) > 1$, and the fact that for $n = 0, 1, 2, \dots$ the gamma function has a simple pole at $-n$ with residue $(-1)^n/n!$, combine with the functional equation to show that the only zeros of the extended $\zeta(s)$ in the left half plane are simple zeros at $s = -2, -4, -6, \dots$. Also, the pole of $Z(s)$ at $s = 0$ shows that that the extended $\zeta(s)$ doesn't vanish at $s = 0$; indeed, because

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \overset{s\to 0}{\sim} \frac{2}{s}\zeta(s)$$

and

$$Z(1-s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \overset{s\to 0}{\sim} \pi^{-1/2}\pi^{1/2}\frac{1}{1-s-1} = -\frac{1}{s},$$

the functional equation says that $\zeta(0) = -1/2$. Another famous result is that because

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \overset{s\to -1}{\sim} \pi^{1/2}(-2\pi^{1/2})\zeta(s) = -2\pi\zeta(s)$$

and

$$Z(1-s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \overset{s\to -1}{\sim} \pi^{-1}\frac{\pi^2}{6} = \frac{\pi}{6},$$

the functional equation says that $\zeta(-1) = -1/12$. These results do not attribute values to the sums $1+1+1+\cdots$ and $1+2+3+\cdots$. More generally, a result that we have established earlier,

$$\zeta(k) = -\frac{(2\pi i)^k}{2\,k!}B_k, \quad k \geq 2 \text{ even},$$

with B_k the kth Bernoulli number, combines with the functional equation to give

$$\zeta(1-k) = -\frac{B_k}{k}, \quad k \geq 2 \text{ even}.$$

This is tidier than the value of $\zeta(k)$, with no power of π and no factorial. For elaborate computations with the zeta function and its variants that have similar functional equations, it is an indispensable gain of ease—and of likely-correct results—to move to the tidy divergent region of the functional equation, work there, and then take the answer back to the region of convergence if so desired.

Part 2. DIRICHLET L-FUNCTIONS: ANALYTIC CONTINUATION AND FUNCTIONAL EQUATION

7. Theta function of a primitive Dirichlet character

A Dirichlet character

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

is called *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

A primitive even Dirichlet character modulo N has an associated theta function

$$\theta_+(\chi, t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \chi(n) e^{-\pi n^2 t/N}, \quad t > 0.$$

The sum $\theta_+(\chi, t)$ is zero for odd χ. A primitive odd Dirichlet character modulo N has an associated theta function

$$\theta_-(\chi, t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} n\chi(n) e^{-\pi n^2 t/N}, \quad t > 0.$$

The sum $\theta_-(\chi, t)$ is zero for even χ. To gather the two cases, associate to any Dirichlet character χ an integer $\delta = \delta(\chi)$,

$$\delta = \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases}$$

Now for a primitive Dirichlet character modulo N, the definition

$$\boxed{\theta(\chi, t) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} n^\delta \chi(n) e^{-\pi n^2 t/N}, \quad t > 0}$$

captures both definitions above. We will derive a modular transformation law for this theta function.

8. A Poisson summation result

Recall that the Poisson summation formula says that for suitable functions $f : \mathbb{R} \longrightarrow \mathbb{C}$,

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \mathcal{F}f(n) e^{2\pi i n x}, \quad x \in \mathbb{R}.$$

Recall also that the Fourier transform of a dilation $f_r(x) = f(xr)$ of a suitable function f is

$$\mathcal{F}(f_r) = r^{-1}(\mathcal{F}f)_{r^{-1}}, \quad r > 0.$$

And recall that the Gaussian function,

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \qquad g(x) = e^{-\pi x^2},$$

is its own Fourier transform, *i.e.*, $\mathcal{F}g = g$.

Using the results just mentioned, compute that for $x \in \mathbb{R}$ and $r > 0$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{-\pi(x+n)^2/r} &= \sum_{n\in\mathbb{Z}} g_{r^{-1/2}}(x+n) \\ &= \sum_{n\in\mathbb{Z}} \mathcal{F}(g_{r^{-1/2}})(n)e^{2\pi inx} && \text{by Poisson summation} \\ &= r^{1/2}\sum_{n\in\mathbb{Z}} (\mathcal{F}g)_{r^{1/2}}(n)e^{2\pi inx} && \text{by the dilation formula} \\ &= r^{1/2}\sum_{n\in\mathbb{Z}} g_{r^{1/2}}(n)e^{2\pi inx} && \text{by the property of the Gaussian} \\ &= r^{1/2}\sum_{n\in\mathbb{Z}} e^{2\pi inx-\pi n^2 r}.\end{aligned}$$

A slight rearrangement gives

$$\sum_{n\in\mathbb{Z}} e^{2\pi inx-\pi n^2 r} = r^{-1/2}\sum_{n\in\mathbb{Z}} e^{-\pi(x+n)^2/r}, \quad x\in\mathbb{R},\ r>0.$$

Differentiate with respect to x to get

$$\sum_{n\in\mathbb{Z}} ne^{2\pi inx-\pi n^2 r} = ir^{-3/2}\sum_{n\in\mathbb{Z}} (x+n)e^{-\pi(x+n)^2/r}, \quad x\in\mathbb{R},\ r>0.$$

Recall the integer δ that is 0 for an even Dirichlet character and 1 for an odd one. This integer lets us gather the previous two displays,

$$\sum_{n\in\mathbb{Z}} n^\delta e^{2\pi inx-\pi n^2 r} = i^\delta r^{-1/2-\delta}\sum_{n\in\mathbb{Z}} (x+n)^\delta e^{-\pi(x+n)^2/r}, \quad x\in\mathbb{R},\ r>0.$$

Although this result bears some resemblance to a modular transformation law for the theta function of a Dirichlet character, to make things dovetail perfectly we also need to consider Gauss sums.

9. Gauss sums of primitive Dirichlet characters

A primitive Dirichlet character

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

has associated Gauss sums

$$\tau_n(\chi) = \sum_{m=0}^{N-1} \chi(m)e^{2\pi inm/N}, \quad n\in\mathbb{Z}.$$

Note that $\tau_n(\chi) = \mathcal{F}\chi(n)$, viewing the basic character of $\mathbb{Z}/N\mathbb{Z}$ as $\psi(x) = e^{-2\pi ix/N}$. Especially, the basic Gauss sum associated to χ is

$$\tau(\chi) = \tau_1(\chi) = \mathcal{F}\chi(1) = \sum_{m=0}^{N-1} \chi(m)e^{2\pi im/N}.$$

The sum $\tau_n(\chi)$ could be taken only over $m \in (\mathbb{Z}/N\mathbb{Z})^\times$, and the next proposition will show that we could consider $\tau_n(\chi)$ only for $n \in (\mathbb{Z}/N\mathbb{Z})^\times$ because otherwise $\tau_n(\chi) = 0$. However, the proof of the main result of this section,

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N,$$

to be established after the proposition, is transparent when we sum over $\mathbb{Z}/N\mathbb{Z}$ rather than summing only over $(\mathbb{Z}/N\mathbb{Z})^\times$. Summing over $\mathbb{Z}/N\mathbb{Z}$ lets us use the fact that exponentiation is an additive character along with χ being a multiplicative character.

Proposition 9.1. *If χ is primitive modulo N then*

$$\overline{\chi}(n)\tau(\chi) = \tau_n(\chi), \quad n \in \mathbb{Z}.$$

Proof. First assume that $\gcd(n, N) = 1$. The relation $\overline{\chi}(n)\chi(n) = 1$ quickly proves the formula,

$$\begin{aligned}\overline{\chi}(n)\tau(\chi) &= \overline{\chi}(n) \sum_{m=0}^{N-1} \chi(m) e^{2\pi i m/N} \\ &= \overline{\chi}(n) \sum_{m=0}^{N-1} \chi(nm) e^{2\pi i n m/N} \\ &= \sum_{m=0}^{N-1} \chi(m) e^{2\pi i n m/N} = \tau_n(\chi).\end{aligned}$$

Now assume that $\gcd(n, N) > 1$. We need to show that $\tau_n(\chi) = 0$. For this argument it is more convenient to rewrite the Gauss sum as

$$\tau_n(\chi) = \sum_{m \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(m) e^{2\pi i n m/N}.$$

The degenerate case $N = 1$ is excluded because $\gcd(n, N) > 1$. Let $g = \gcd(n, N)$, so that $n = n'g$ and $N = N'g$ for some integer n' and positive integer N'. The surjection

$$(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/N'\mathbb{Z})^\times$$

has kernel

$$K = \{k \in (\mathbb{Z}/N\mathbb{Z})^\times : k = 1 \bmod N'\},$$

and thus $(\mathbb{Z}/N\mathbb{Z})^\times$ has a coset decomposition

$$(\mathbb{Z}/N\mathbb{Z})^\times = \bigsqcup_r rK,$$

where the representatives $r \in (\mathbb{Z}/N\mathbb{Z})^\times$ take distinct values modulo N'. All elements m of a given coset rK satisfy $m = r \bmod N'$. Note also that

$$e^{2\pi i n m/N} = e^{2\pi i n' m/N'} = e^{2\pi i n' r/N'} \quad \text{for } m \in rK,$$

and this value depends only on r. Thus altogether we have

$$\tau_n(\chi) = \sum_r \sum_{m \in rK} \chi(m) e^{2\pi i n m/N} = \sum_r \chi(r) e^{2\pi i n' r/N'} \sum_{k \in K} \chi(k).$$

Now we use the fact that χ is primitive. Specifically, χ doesn't factor through the quotient $(\mathbb{Z}/N'\mathbb{Z})^\times \approx (\mathbb{Z}/N\mathbb{Z})^\times/K$ of $(\mathbb{Z}/N\mathbb{Z})^\times$, so it isn't identically 1 on K. Consequently the inner sum at the end of the previous display is 0, showing that $\tau_n(\chi) = 0$ as desired. $\square$

Now compute, using the proposition's result $\overline{\chi}(n)\tau(\chi) = \tau_n(\chi)$ for the second equality,

$$\begin{aligned}\tau(\chi)\tau(\overline{\chi}) &= \sum_{n=0}^{N-1} \overline{\chi}(n)\tau(\chi)e^{2\pi in/N} = \sum_{n=0}^{N-1} \tau_n(\chi)e^{2\pi in/N} \\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \chi(m)e^{2\pi inm/N}e^{2\pi in/N} \\ &= \sum_{m=0}^{N-1} \chi(m) \sum_{n=0}^{N-1} e^{2\pi i(m+1)n/N} = \chi(-1)N,\end{aligned}$$

the last equality holding because the inner sum is N when $m = -1$ and 0 otherwise. Because $\tau(\overline{\chi}) = \chi(-1)\overline{\tau(\chi)}$, as is readily verified, the previous display shows that $|\tau(\chi)| = N^{1/2}$.

Our proof that $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N$ in an earlier writeup was simpler because it could use its circumstance that N is prime, giving $\mathbb{Z}/N\mathbb{Z} = (\mathbb{Z}/N\mathbb{Z})^\times \cup \{0\}$ and making every nontrivial character modulo N primitive. But now N needn't be prime.

Recall that a Dirichlet character χ is even if $\chi(-1) = 1$ and odd if $\chi(-1) = -1$, and recall that we set the integer δ to 0 for an even Dirichlet character and to 1 for an odd Dirichlet character. Thus $\chi(-1) = (-1)^\delta$. Introduce the *root number* of a primitive Dirichlet character, a complex number of absolute value 1,

$$W(\chi) = \frac{\tau(\chi)}{i^\delta N^{1/2}}\,.$$

The root number is chosen so that regardless of the parity of χ, the relation $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N$ becomes $W(\chi)W(\overline{\chi}) = 1$, or

$$W(\overline{\chi}) = W(\chi)^{-1}.$$

10. Dirichlet theta function transformation law

Recall that the theta function of a primitive Dirichlet character χ modulo N is

$$\theta(\chi, t) = \sum_{n\in\mathbb{Z}} n^\delta\chi(n)e^{-\pi n^2 t/N}, \quad t > 0.$$

Compute, using the identity $\overline{\chi}(n)\tau(\chi) = \tau_n(\chi)$ for the second equality,

$$\begin{aligned}\tau(\chi)\,\theta(\overline{\chi}, t) &= \sum_{n\in\mathbb{Z}} \overline{\chi}(n)\tau(\chi)n^\delta e^{-\pi n^2 t/N} = \sum_{n\in\mathbb{Z}} \tau_n(\chi)n^\delta e^{-\pi n^2 t/N} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m=0}^{N-1} \chi(m)e^{2\pi inm/N}n^\delta e^{-\pi n^2 t/N} \\ &= \sum_{m=0}^{N-1} \chi(m) \sum_{n\in\mathbb{Z}} n^\delta e^{2\pi inm/N - \pi n^2 t/N}.\end{aligned}$$

Apply the relation from Poisson summation,

$$\sum_{n\in\mathbb{Z}} n^\delta e^{2\pi inx - \pi n^2 r} = i^\delta r^{-1/2-\delta} \sum_{n\in\mathbb{Z}} (x+n)^\delta e^{-\pi(x+n)^2/r},$$

with $x = m/N$ and $r = t/N$,

$$\begin{aligned}\tau(\chi)\,\theta(\overline{\chi},t) &= i^\delta (N/t)^{1/2+\delta}\sum_{m=0}^{N-1}\chi(m)\sum_{n\in\mathbb{Z}}(m/N+n)^\delta e^{-\pi(m/N+n)^2N/t}\\ &= i^\delta N^{1/2}t^{-1/2-\delta}\sum_{m=0}^{N-1}\chi(m)\sum_{n\in\mathbb{Z}}(m+nN)^\delta e^{-\pi(m+nN)^2(1/t)/N}\\ &= i^\delta N^{1/2}t^{-1/2-\delta}\sum_{\ell\in\mathbb{Z}}\ell^\delta\chi(\ell)e^{-\pi\ell^2(1/t)/N}\\ &= i^\delta N^{1/2}t^{-1/2-\delta}\,\theta(\chi,1/t).\end{aligned}$$

A slight rearrangement gives the modular transformation law,

$$\boxed{\theta(\chi,1/t) = W(\chi)\,t^{1/2+\delta}\,\theta(\overline{\chi},t),\quad t>0.}$$

11. Functional equation

Let χ be a nontrivial primitive Dirichlet character. Recall that its Dirichlet L-function is

$$L(\chi,s) = \sum_{n\ge 1}\chi(n)n^{-s},\quad \mathrm{Re}(s)>1.$$

Let N be the conductor of χ. Recall that the integer δ is 0 or 1 depending whether χ is even or odd. Because χ is nontrivial, its conductor N is greater than 1, and so $\chi(0)=0$. Therefore, for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$,

$$\begin{aligned}\frac{1}{2}\int_{t=0}^\infty \theta(\chi,t)t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t} &= \sum_{n\ge1}n^\delta\chi(n)\int_{t=0}^\infty e^{-\pi n^2t/N}t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t}\\ &= \sum_{n\ge1}n^\delta\chi(n)(\pi n^2/N)^{-(s+\delta)/2}\Gamma((s+\delta)/2)\\ &= (\pi/N)^{-(s+\delta)/2}\Gamma((s+\delta)/2)L(\chi,s).\end{aligned}$$

That is, the completed Dirichlet L-function

$$\boxed{\Lambda(\chi,s) \stackrel{\mathrm{def}}{=} (\pi/N)^{-(s+\delta)/2}\Gamma((s+\delta)/2)L(\chi,s),\quad \mathrm{Re}(s)>1}$$

has an integral representation as the Mellin transform of the theta function,

$$\boxed{\Lambda(\chi,s) = \frac{1}{2}\int_{t=0}^\infty \theta(\chi,t)t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t},\quad \mathrm{Re}(s)>1.}$$

The integral converges at $t=\infty$ independently of the value of s. Compute, using the modular transformation law $\theta(\chi,1/t) = W(\chi)\,t^{1/2+\delta}\,\theta(\overline{\chi},t)$ for the third equality, that the integral is

$$\begin{aligned}\int_{t=0}^\infty \theta(\chi,t)t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t} &= \int_{t=1}^\infty \theta(\chi,t)t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t} + \int_{t=0}^1 \theta(\chi,t)t^{(s+\delta)/2}\,\frac{\mathrm{d}t}{t}\\ &= \int_{t=1}^\infty \left(\theta(\chi,t)t^{(s+\delta)/2} + \theta(\chi,1/t)t^{-(s+\delta)/2}\right)\frac{\mathrm{d}t}{t}\\ &= \int_{t=1}^\infty \left(\theta(\chi,t)t^{(s+\delta)/2} + W(\chi)\,\theta(\overline{\chi},t)t^{(1-s+\delta)/2}\right)\frac{\mathrm{d}t}{t}\,.\end{aligned}$$

The last integral is an entire function of s. Thus the function $\Lambda(\chi, s)$, which is initially a Mellin transform, extends to an entire function of s, also defined as an integral,

$$\boxed{\Lambda(\chi, s) = \frac{1}{2}\int_{t=1}^{\infty} \big(\theta(\chi, t)t^{s/2} + W(\chi)\,\theta(\overline{\chi}, t)t^{(1-s)/2}\big)t^{\delta/2}\,\frac{\mathrm{d}t}{t}\,, \quad s \in \mathbb{C}.}$$

Consequently $L(\chi, s)$ extends to an entire function of s as well. Furthermore, because $W(\overline{\chi}) = W(\chi)^{-1}$, replacing s by $1 - s$ and χ by $\overline{\chi}$ in the last integral multiplies it by $W(\chi)^{-1}$,

$$\begin{aligned}\Lambda(\overline{\chi}, 1-s) &= \int_{t=1}^{\infty} \big(\theta(\overline{\chi}, t)t^{(1-s)/2} + W(\overline{\chi})\,\theta(\chi, t)t^{s/2}\big)t^{\delta/2}\,\frac{\mathrm{d}t}{t}\\ &= W(\chi)^{-1}\int_{t=1}^{\infty} \big(W(\chi)\,\theta(\overline{\chi}, t)t^{(1-s)/2} + \theta(\chi, t)t^{s/2}\big)t^{\delta/2}\,\frac{\mathrm{d}t}{t}\,.\end{aligned}$$

Therefore, recognizing the last integral as the boxed integral just above, we have the functional equation

$$\boxed{W(\chi)\,\Lambda(\overline{\chi}, 1-s) = \Lambda(\chi, s), \quad s \in \mathbb{C}.}$$

When χ is even, the Euler product of $L(\chi, s)$ for $\mathrm{Re}(s) > 1$ combines with the functional equation to show that the only zeros of the extended $L(\chi, s)$ in the left half plane are simple zeros at $s = -2, -4, -6, \dots$. The functional equation also shows that $L(\chi, s)$ has a zero at $s = 0$, and the nontrivial fact that Dirichlet L-functions don't vanish at $s = 1$ shows that the zero at $s = 0$ is simple.

When χ is odd, the Euler product of $L(\chi, s)$ for $\mathrm{Re}(s) > 1$ combines with the functional equation to show that the only zeros of the extended $L(\chi, s)$ in the left half plane are simple zeros at $s = -1, -3, -5, \dots$, and the fact that Dirichlet L-functions don't vanish at $s = 1$ shows that $L(0, \chi)$ is nonzero.

12. Quadratic root numbers

Let F be a quadratic number field. Its Dedekind zeta function,

$$\zeta_F(s) = \sum_{\mathfrak{a}} \mathrm{N}\mathfrak{a}^{-s}, \quad \mathrm{Re}(s) > 1,$$

has a completion $Z_F(s)$ that extends meromorphically to $\mathbb{C}$ with simple poles at $s = 0, 1$ and satisfies the functional equation $Z_F(s) = Z_F(1-s)$. The quadratic number field F has an associated quadratic character $\chi = \chi_F$ whose conductor is the absolute discriminant of F. The arithmetic of the quadratic field encodes as the identity $Z_F(s) = Z_{\mathbb{Q}}(s)\Lambda(\chi, s)$ where $Z_{\mathbb{Q}}$ is the completed Euler–Riemann zeta function. Noting that $\overline{\chi} = \chi$ because χ is quadratic, compute that

$$\begin{aligned}Z_F(1-s) &= Z_F(s) && \text{by the functional eqn for } Z_F\\ &= Z_{\mathbb{Q}}(s)\Lambda(\chi, s) && \text{factoring } Z_F\\ &= Z_{\mathbb{Q}}(1-s)W(\chi)\Lambda(\chi, 1-s) && \text{by the functional eqns for } Z_{\mathbb{Q}} \text{ and } \Lambda\\ &= W(\chi)Z_F(1-s) && \text{regathering } Z_F.\end{aligned}$$

Thus $W(\chi) = 1$ for the quadratic character χ. In particular, with $\chi(n) = (n/p)$ for an odd prime p, this result captures the value of the quadratic Gauss sum, $\tau(\chi) = p^{1/2}$ if $p \equiv 1 \bmod 4$ and $\tau(\chi) = ip^{1/2}$ if $p \equiv 3 \bmod 4$.