EXPONENTIAL AND LOGARITHMIC POWER SERIES
FORMAL PROPERTIES

This writeup shows, purely formally, with no reference to analysis, that the power series definitions

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \text{and} \qquad \ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

satisfy the properties (noting that $1-(x+y-xy) = (1-x)(1-y)$ for the second and that $1-(1-\exp(x)) = \exp(x)$ for the fourth)

$$\begin{aligned}
\exp(x+y) &= \exp(x)\exp(y) \\
\ln\bigl(1-(x+y-xy)\bigr) &= \ln(1-x) + \ln(1-y) \\
\exp(\ln(1-x)) &= 1-x \\
\ln\bigl(1-(1-\exp(x))\bigr) &= x
\end{aligned}$$

so long as we assume that we are working in characteristic 0.

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1. Exponential property

Immediately from the binomial theorem,

$$\begin{aligned}
\exp(x)\exp(y) &= \sum_{j=0}^{\infty} \frac{1}{j!} x^j \sum_{k=0}^{\infty} \frac{1}{k!} y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{j,k \geq 0 \\ j+k=n}} \frac{n!}{j!\,k!} x^j y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n \\
&= \exp(x+y).
\end{aligned}$$

2. Logarithmic property

Granting that the exponential and the logarithm are inverses, as will be shown below, compute, using $\exp\circ\ln = 1$ at the first step, $\exp(z)\exp(w) = \exp(z+w)$ at the second, and $\ln\circ\exp = 1$ at the third,

$$\begin{aligned}\ln\big((1-x)(1-y)\big) &= \ln\big(\exp(\ln(1-x))\exp(\ln(1-y))\big)\\ &= \ln(\exp\big(\ln(1-x)+\ln(1-y)\big))\\ &= \ln(1-x)+\ln(1-y).\end{aligned}$$

However, we also show this result directly. The sum of two logarithmic power series is

$$\ln(1-x)+\ln(1-y) = -\sum_{n=1}^{\infty}\frac{1}{n}(x^n+y^n).$$

And the logarithmic power series of the product is

$$\ln((1-x)(1-y)) = \ln(1-(x+y-xy)) = -\sum_{n=1}^{\infty}\frac{1}{n}(x+y-xy)^n.$$

By the trinomial theorem, this is

$$\begin{aligned}\ln((1-x)(1-y)) &= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\substack{j,k,\ell\geq 0\\ j+k+\ell=n}}\binom{n}{j,k,\ell}(-1)^{\ell}x^{j+\ell}y^{k+\ell}\\ &= -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\substack{j,k\geq 0\\ j+k\leq n}}(-1)^{n-j-k}\frac{n!}{j!\,k!\,(n-j-k)!}x^{n-k}y^{n-j}.\end{aligned}$$

The terms where $(j,k)=(n,0)$ or $(j,k)=(0,n)$ give $-\sum_{n=1}^{\infty}\frac{1}{n}(x^n+y^n)$, which we recognize from above as $\ln(1-x)+\ln(1-y)$. So what needs to be shown is that the rest of $\ln((1-x)(1-y))$ is 0. This is the condition that for every pair (α,β) of positive integers, the coefficient of $x^{\alpha}y^{\beta}$ vanishes, and we may take $\alpha\leq\beta$ because the whole situation is symmetric in x and y,

$$\sum_{n=1}^{\infty}\sum_{\substack{0\leq j,k<n\\ j+k\leq n\\ n-k=\alpha\\ n-j=\beta}}(-1)^{n-j-k}\frac{(n-1)!}{j!\,k!\,(n-j-k)!} = 0,\quad 0<\alpha\leq\beta.$$

We see that $j=n-\beta$ and $k=n-\alpha$, so that $0\leq n-\alpha$ and $0\leq n-\beta$, i.e., $n\geq\alpha$ and $n\geq\beta$ (and the second of these implies the first because $\alpha\leq\beta$) and the summation condition $j+k\leq n$ is $n\leq\alpha+\beta$. So now the needed condition is that infinitely many finite alternating sums vanish as follows,

$$\sum_{n=\beta}^{\alpha+\beta}(-1)^{\alpha+\beta-n}\frac{(n-1)!}{(n-\alpha)!\,(n-\beta)!\,(\alpha+\beta-n)!} = 0,\quad 0<\alpha\leq\beta.$$

Let $k=\alpha+\beta-n$, which varies from 0 to α, and the needed condition becomes

$$\sum_{k=0}^{\alpha}(-1)^k\frac{(\alpha-1+\beta-k)!}{(\beta-k)!\,(\alpha-k)!\,k!} = 0,\quad 0<\alpha\leq\beta.$$

The unsigned summand is

$$\frac{(\alpha-1)!}{\alpha!}\cdot\frac{\alpha!}{k!\,(\alpha-k)!}\cdot\frac{(\alpha-1+\beta-k)!}{(\alpha-1)!\,(\beta-k)!}=\frac{1}{\alpha}\binom{\alpha}{k}\binom{\alpha-1+\beta-k}{\alpha-1}.$$

And so, because $1/\alpha$ is constant as k varies, the needed condition is combinatorial,

$$\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k}\binom{\alpha-1+\beta-k}{\alpha-1}=0,\quad 0<\alpha\le\beta.$$

We argue combinatorially that indeed the sum in the previous display is 0. For any $k\in\{0,\dots,\alpha\}$ and any size-k subset A_k of $\{1,\dots,\alpha\}$, let S_{A_k} denote the set of $(\alpha-1)$-combinations (unordered $(\alpha-1)$-element sets) from $\{1,\dots,\alpha-1+\beta\}-A_k$. Thus

- $S_{A_o}=S_\emptyset$ consists of the $(\alpha-1)$-combinations from $\{1,\dots,\alpha-1+\beta\}$ that needn't omit any element of $\{1,\dots,\alpha\}$ but may do so
- each S_{A_1} consists of the $(\alpha-1)$-combinations that omit the lone element of A_1 and possibly omit other elements of $\{1,\dots,\alpha\}$ as well
- each S_{A_2} consists of the $(\alpha-1)$-combinations that omit the two elements of A_2 and possibly other elements of $\{1,\dots,\alpha\}$ as well

and so on. Inclusion-exclusion counting says that the sum of $(-1)^{|A_k|}|S_{A_k}|$ over all such k and A_k counts the $(\alpha-1)$-combinations from $\{1,\dots,\alpha-1+\beta\}$ that omit *none* of $\{1,\dots,\alpha\}$, i.e., contain them all. Because $\alpha>\alpha-1$ there are no such combinations and so the sum vanishes,

$$\sum_{k=0}^{\alpha}(-1)^k\sum_{A_k}|S_{A_k}|=0,\quad 0<\alpha\le\beta.$$

For each k there are $\binom{\alpha}{k}$ sets A_k, for each of which $|S_{A_k}|=\binom{\alpha-1+\beta-k}{\alpha-1}$, and so the previous display is exactly what we need,

$$\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k}\binom{\alpha-1+\beta-k}{\alpha-1}=0,\quad 0<\alpha\le\beta.$$

This completes the proof that the power series of $\ln(1-x)+\ln(1-y)$ and of $\ln(1-(x+y-xy))$ are formally equal.

(A slightly variant proof uses a more set theoretic statement of the inclusion-exclusion principle, as follows. For any finite collection of finite sets, $\{S_i : i\in I\}$, and with the empty intersection $\bigcap_{j\in\emptyset}S_j$ understood to be $\bigcup_{i\in I}S_i$,

$$\sum_{J\subset I}(-1)^{|J|}\Big|\bigcap_{j\in J}S_j\Big|=0.$$

This equality holds by induction on $|I|$, as the reader is invited to show. Especially, if $\left|\bigcap_{j\in J}S_j\right|$ depends only on $|J|$—call it $f(|J|)$—then the previous display simplifies to

$$\sum_{k=0}^{|I|}(-1)^k\binom{|I|}{k}f(k)=0.$$

Now let $I=\{1,\dots,\alpha\}$, and for each $i\in I$ let S_i be the set of $(\alpha-1)$-combinations of $\{1,\dots,\alpha-1+\beta\}$ that omit i. Thus $|I|=\alpha$ and $f(k)=\binom{\alpha-1+\beta-k}{\alpha-1}$, and so as above,

$$\sum_{k=0}^{\alpha}(-1)^k\binom{\alpha}{k}\binom{\alpha-1+\beta-k}{\alpha-1}=0.$$

Whereas the previous paragraph counted out the solution to the specific problem at hand, this paragraph has counted out a general inclusion-exclusion principle that is broadly useful and then specialized it to solve the problem.)

3. Composition of formal power series

To show that the exponential and logarithmic power series are inverses, we first discuss composition of power series in general.

Consider two power series, both having constant term 0,

$$a(x)=\sum_{n\geq 1}a_nx^n \qquad b(x)=\sum_{m\geq 1}b_mx^m.$$

In what follows, indices named n or m are understood to be at least 1. For any n, the nth power of $b(x)$ is

$$(b(x))^n=\sum_{m_1,\dots,m_n}b_{m_1}\cdots b_{m_n}x^{m_1+\cdots+m_n}$$

Introduce notation for the length and the size of any vector $\vec{m}=(m_1,\dots,m_n)$,

$$\ell(\vec{m})=n \qquad |\vec{m}|=m_1+\cdots+m_n,$$

and introduce an abbreviation of a product,

$$b_{\vec{m}}=b_{m_1}\cdots b_{m_{\ell(\vec{m})}}.$$

So now, concisely,

$$(b(x))^n=\sum_{\vec{m}:\ell(\vec{m})=n}b_{\vec{m}}x^{|\vec{m}|},$$

and so $a(b(x))=\sum_{n\geq 1}a_n\sum_{\vec{m}:\ell(\vec{m})=n}b_{\vec{m}}x^{|\vec{m}|}=\sum_{\vec{m}}a_{\ell(\vec{m})}b_{\vec{m}}x^{|\vec{m}|}$ is therefore

$$a(b(x))=\sum_{k\geq 1}c_kx^k \quad \text{where } c_k=\sum_{|\vec{m}|=k}a_{\ell(\vec{m})}b_{\vec{m}}.$$

More handily,

$$a(b(x))=\sum_{k\geq 1}c_kx^k \quad \text{where } c_k=\sum_{n=1}^{k}a_n\sum_{\substack{\ell(\vec{m})=n\\ |\vec{m}|=k}}b_{\vec{m}}. \tag{1}$$

For example,

$$\begin{aligned}
c_1&=a_1b_{(1)}=a_1b_1\\
c_2&=a_1b_{(2)}+a_2b_{(1,1)}=a_1b_2+a_2b_1^2\\
c_3&=a_1b_{(3)}+a_2(b_{(1,2)}+b_{(2,1)})+a_3b_{(1,1,1)}=a_1b_3+2a_2b_1b_2+a_3b_1^3.
\end{aligned}$$

4. The exponential inverts the logarithm

For $a(x) = \exp(x) - 1$ and $b(x) = \ln(1-x)$, so that $a(b(x)) = \exp(\ln(1-x)) - 1$, which we want to be $-x$, we have $a_n = 1/n!$ and $b_m = -1/m$ and so (1) gives $\exp(\ln(1-x)) - 1 = \sum_{k\geq 1} c_k x^k$ where

$$c_k = \sum_{n=1}^{k} \frac{(-1)^n}{n!} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1 \cdots m_k}\,.$$

So $c_1 = -1/1! \cdot 1/1 = -1$ and $c_2 = -1/1! \cdot 1/2 + 1/2! \cdot 1/(1 \cdot 1) = 0$. To show that $\exp(\ln(1-x)) - 1 = -x$ as formal power series, i.e., that $c_k = 0$ for $k \geq 2$, it suffices to show that

$$\sum_{n=1}^{k} (-1)^n \frac{k!}{n!} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1 \cdots m_n} = 0, \quad k \geq 2.$$

The unsigned summand $k!/n! \sum_{\vec{m}} 1/(m_1 \cdots m_n)$ counts how many permutations in S_k decompose into n cycles, including trivial cycles of one element. Indeed, there are $k!$ ways to write 1 through k left to right, and then for each of them, $\sum_{m_1+\cdots+m_n=k} 1$ ways to parenthesize in order to create n cycles, and for each $(m_1, \ldots, m_n)$ we must divide by $m_1 \cdots m_n$ to account for rewriting the cycles, and we must divide by $n!$ to account for rewriting their product. Further, left multiplication by $(1\,2)$ has the effect

$$(1\, a_2\, \ldots\, a_{c-1}\, 2\, a_{c+1}\, \ldots\, a_d) \longleftrightarrow (1\, a_2\, \ldots\, a_{c-1})(2\, a_{c+1}\, \ldots\, a_d),$$

so it bijects between the elements of S_k that have an odd number of cycles and those that have an even number. Hence the alternating sum is 0.

5. The logarithm inverts the exponential

Similarly, with $a(x) = \ln(1-x)$ and $b(x) = 1 - \exp(x)$, so that $a(b(x)) = \ln(\exp(x))$, which we want to be x, now $a_n = -1/n$ for $n \geq 1$ and $b_n = 1/n!$ for $n \geq 1$. This time (1) gives $\ln(\exp(x)) = \sum_{k\geq 1} c_k x^k$ where

$$c_k = \sum_{n=1}^{k} \frac{(-1)^n}{n} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_k!}\,.$$

So $c_1 = 1/1 \cdot 1/1! = 1$ and $c_2 = 1/1 \cdot 1/2! - 1/2 \cdot 1/(1! \cdot 1!) = 0$. To show that $\ln(\exp(x)) = x$ as formal power series, i.e., that $c_k = 0$ for $k \geq 2$, it suffices to show that

$$\sum_{n=1}^{k} (-1)^n \frac{k!}{n} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_n!} = 0, \quad k \geq 2.$$

This time the unsigned summand $k!/n \sum_{\vec{m}} 1/(m_1!...m_n!)$ counts how many ways $\{1, \ldots, k\}$ can be written as a *cycle* of n nonempty subsets. The map

$$\{1\}\{\ldots\} \cdots \longleftrightarrow \{1, \ldots\} \cdots$$

bijects the even such cycles and the odd such cycles. Hence the alternating sum is 0.