THE EXPONENTIAL AND LOGARITHMIC POWER SERIES ARE FORMAL INVERSES

1. Formal composition of power series

Consider two power series, both having constant term 0,

$$a(x) = \sum_{n\geq 1} a_n x^n \qquad\qquad b(x) = \sum_{m\geq 1} b_m x^m.$$

In what follows, indices named n or m are understood to be at least 1. For any n,

$$(b(x))^n = \sum_{m_1,\dots,m_n} b_{m_1}\cdots b_{m_n} x^{m_1+\cdots+m_n}$$

Introduce notation for the length and the size of any vector $\vec{m} = (m_1,\dots,m_n)$,

$$\ell(\vec{m}) = n \qquad\qquad |\vec{m}| = m_1+\cdots+m_n,$$

and introduce an abbreviation of a product,

$$b_{\vec{m}} = b_{m_1}\cdots b_{m_{\ell(\vec{m})}}.$$

So now, concisely,

$$(b(x))^n = \sum_{\vec{m}:\ell(\vec{m})=n} b_{\vec{m}} x^{|\vec{m}|},$$

and so $a(b(x)) = \sum_{n\geq 1} a_n \sum_{\vec{m}:\ell(\vec{m})=n} b_{\vec{m}} x^{|\vec{m}|} = \sum_{\vec{m}} a_{\ell(\vec{m})} b_{\vec{m}} x^{|\vec{m}|}$ is therefore

$$a(b(x)) = \sum_{k\geq 1} c_k x^k \quad \text{where } c_k = \sum_{|\vec{m}|=k} a_{\ell(\vec{m})} b_{\vec{m}}.$$

More handily,

$$a(b(x)) = \sum_{k\geq 1} c_k x^k \quad \text{where } c_k = \sum_{n=1}^{k} a_n \sum_{\substack{\ell(\vec{m})=n\\ |\vec{m}|=k}} b_{\vec{m}}. \tag{1}$$

For example, $c_1 = a_1 b_{(1)} = a_1 b_1$; $c_2 = a_1 b_{(2)} + a_2 b_{(1,1)} = a_1 b_2 + a_2 b_1^2$; $c_3 = a_1 b_{(3)} + a_2(b_{(1,2)} + b_{(2,1)}) + a_3 b_{(1,1,1)} = a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3$.

2. The exponential inverts the logarithm

For $a(x) = \exp(x) - 1$ and $b(x) = \ln(1-x)$, so that $a(b(x)) = -x$, we have $a_n = 1/n!$ and $b_m = -1/m$ and so (1) gives $\exp(\ln(1-x)) - 1 = \sum_{k\geq 1} c_k x^k$ where

$$c_k = \sum_{n=1}^{k} \frac{(-1)^n}{n!} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1\cdots m_k}.$$

So $c_1 = -1/1! \cdot 1/1 = -1$ and $c_2 = -1/1! \cdot 1/2 + 1/2! \cdot 1/(1 \cdot 1) = 0$. To show that $\exp(\ln(1-x)) - 1 = -x$ at the level of formal power series, i.e., that $c_k = 0$ for $k \geq 2$, it suffices to show that

$$\sum_{n=1}^{k} (-1)^n \frac{k!}{n!} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1 \cdots m_n} = 0, \quad k \geq 2.$$

The unsigned summand $k!/n! \sum_{\vec{m}} 1/(m_1 \cdots m_n)$ counts how many permutations in S_k decompose into n cycles, including trivial cycles of one element. Indeed, there are $k!$ ways to write 1 through k left to right, and then for each of them, $\sum_{m_1+\cdots+m_n=k} 1$ ways to parenthesize to create n cycles, and for each $(m_1, \ldots, m_n)$ we must divide by $m_1 \cdots m_k$ to avoid writing the same permutation more than once. Further, left multiplication by $(1\,2)$ has the effect

$$(1\, a_2\, \ldots\, a_{c-1}\, 2\, a_{c+1}\, \ldots\, a_d) \longleftrightarrow (1\, a_2\, \ldots\, a_{c-1})(2\, a_{c+1}\, \ldots\, a_d),$$

so it bijects between the elements of S_k that have an odd number of cycles and those that have an even number. Hence the alternating sum is 0.

3. The logarithm inverts the exponential

Similarly, with $a(x) = \ln(1-x)$ and $b(x) = 1 - \exp(x)$, so that $a(b(x)) = \ln(\exp(x))$, now $a_n = -1/n$ for $n \geq 1$ and $b_n = 1/n!$ for $n \geq 1$. This time (1) gives $\ln(\exp(x)) = \sum_{k \geq 1} c_k x^k$ where

$$c_k = \sum_{n=1}^{k} \frac{(-1)^n}{n} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_k!}.$$

So $c_1 = 1/1 \cdot 1/1! = 1$ and $c_2 = 1/1 \cdot 1/2! - 1/2 \cdot 1/(1! \cdot 1!) = 0$. To show that $\ln(\exp(x)) = x$ at the level of formal power series, i.e,. that $c_k = 0$ for $k \geq 2$, it suffices to show that

$$\sum_{n=1}^{k} (-1)^n \frac{k!}{n} \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_n!} = 0, \quad k \geq 2.$$

The unsigned summand $k!/n \sum_{\vec{m}} 1/(m_1!...m_m!)$ counts how many ways $\{1, \ldots, k\}$ can be broken into a *cycle* of n nonempty subsets. The map

$$\{1\}\{\ldots\}\cdots \longleftrightarrow \{1, \ldots\}\ldots$$

bijects the even such cycles and the odd such cycles. Hence the alternating sum is 0.