NUMBER FIELD ZETA INTEGRALS AND L-FUNCTIONS

The basic zeta function is

$$\zeta(s) = \sum_{n\geq 1} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

Riemann proved that the completed basic zeta function,

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1,$$

has an entire meromorphic continuation that satisfies the functional equation

$$Z(1-s) = Z(s), \quad s \in \mathbb{C}.$$

One of his proofs proceeds by arguing as follows.

- Poisson summation shows that the theta function,

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi i n^2 t}, \quad t > 0,$$

  is a modular form,

$$\theta(1/t) = t^{1/2}\theta(t).$$

- Z is the Mellin transform (essentially a Fourier transform) of θ,

$$Z(s) = \tfrac{1}{2}\int_{t>0} (\theta(t)-1)t^{s/2}\,\frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 1.$$

  As $t \to \infty$ the integrand decays rapidly regardless of the value of s, posing no obstacle to the integral's convergence at its improper upper limit of integration. But as $t \to 0^+$ the integrand is $\mathcal{O}(t^{(\mathrm{Re}(s)-1)/2}\,dt/t)$, requiring $\mathrm{Re}(s) > 1$ for the integral to converge at its improper lower limit.
- The transformation law for θ shows that in fact

$$Z(s) = \tfrac{1}{2}\int_{t>1} (\theta(t)-1)(t^{s/2} + t^{(1-s)/2})\,\frac{\mathrm{d}t}{t} - \frac{1}{s} - \frac{1}{1-s}.$$

  With the problematic lower limit of integration no longer present, the right side is now sensible as a meromorphic function for all $s \in \mathbb{C}$, and it visibly satisfies the functional equation.

Similar methods apply to any Dirichlet L-function,

$$L(s,\chi) = \sum_{n\geq 1} \chi(n)n^{-s}, \quad \mathrm{Re}(s) > 1.$$

The Dirichlet L-function has a suitable completion

$$\Lambda(s,\chi), \quad \mathrm{Re}(s) > 1,$$

involving a factor similar to the factor $\pi^{-s/2}\Gamma(s/2)$ for the basic zeta function, but now taking into account the parity and the conductor of χ. Again the completion has a continuation and a functional equation. The calculation now involves a Gauss sum along with the other ingredients, and the functional equation includes a factor called a root number,

$$W(\chi)\Lambda(1-s,\overline{\chi}) = \Lambda(s,\chi).$$

If the Dirichlet character is quadratic then the root number $W(\chi)$ is 1.

Both the basic zeta function and the Dirichlet L-function are defined over the basic number field $\mathbb{Q}$. The technique of completion, continuation, and functional equation apply more generally to the Dedekind zeta function of a number field $\mathbf{k}$,

$$\zeta_{\mathbf{k}}(s) = \sum_{\mathfrak{a}} \mathrm{N}\mathfrak{a}^{-s},$$

and even for any L-function associated to a number field $\mathbf{k}$ and a so-called Hecke character of $\mathbf{k}$,

$$L_{\mathbf{k}}(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s}.$$

But the definition of a Hecke character is complicated, and the calculations and the root number become ever more elaborate as the environment grows.

As Artin, Iwasawa, Tate, and perhaps others knew around 1950, working in the environment of the adeles simplifies the calculation decisively. In the adelic environment, the definition of a Hecke character is simple and natural, and the root number emerges naturally from local calculations.

After defining the *adelic zeta integral* and looking at some of its local factors, these notes establish a global adelic version of its continuation and functional equation. Strikingly, the procedure is essentially identical to Riemann's original argument. The exposition of this material is based on a presentation and written materials by Paul Garrett. Then the notes continue to discuss the local factors of the zeta integral, following Kudla's lecture in the *Introduction to the Langlands Program* volume.

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Part 1. The Zeta Integral

1. Definition of the Zeta Integral

Let $\mathbf{k}$ be a number field, and let $\mathbb{A}$ be its adele ring. Recall that $\mathbf{k}$ is discrete in $\mathbb{A}$ and the quotient $\mathbb{A}/\mathbf{k}$ is compact. Let $\mathbb{J}$ be the idele group of $\mathbf{k}$. Recall that $\mathbf{k}^\times$ is discrete in the *unit* idele group $\mathbb{J}^1$ and the quotient $\mathbb{J}^1/\mathbf{k}^\times$ is compact. (The statement here is *Fujisaki's Lemma*, encompassing both the structure theorem for the units group of the integers of $\mathbf{k}$ and finiteness of the class number of $\mathbf{k}$. The adelic repackaging of those results as Fujisaki's Lemma is optimal for our purposes here.)

Let s be a complex parameter, let

$$\chi : \mathbb{J} \longrightarrow \mathbb{C}^\times$$

be a Hecke character of $\mathbb{J}$, and let

$$\varphi : \mathbb{A} \longrightarrow \mathbb{C}$$

be an adelic Schwartz function. Let $|\ |$ denote the idele norm, and let $d^\times$ denote Haar measure on $\mathbb{J}$. Then the *adelic zeta integral* associated to χ and φ is

$$\boxed{Z(s,\chi,\varphi) = \int_{\mathbb{J}} |\alpha|^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha.}$$

Here s and χ combine to encode the overall character $\chi|\cdot|^s$ being integrated against the Schwartz function. Since χ need not be discretely parametrized, the parameters of the zeta integral thus incorporate some redundancy:

$$Z(s'+s,\chi,\varphi) = Z(s',\chi|\cdot|^s,\varphi) \quad \text{for all } s', s, \chi.$$

The integral converges for values of s in a right half-plane.

2. Local Zeta Integrals

The Hecke character in the zeta integral decomposes as a product of local characters,

$$\chi = \bigotimes_v \chi_v : \mathbb{J} \longrightarrow \mathbb{C}^\times.$$

The assertion that the Schwartz function in the zeta integral decomposes similarly,

$$\varphi = \bigotimes_v \varphi_v, \quad \text{each } \varphi_v \text{ is Schwartz},$$

is not quite true, since in fact φ is a finite sum of such products, but we treat φ as a monomial from now on without further comment. For a nonarchimedean place v, the local Schwartz space consists of the compactly supported locally constant functions on $\mathbf{k}_v$, and for an archimedean place v the local Schwartz space consists of the smooth functions on $\mathbf{k}_v$ all of whose derivatives are rapidly decreasing.

In consequence of χ and φ factoring, the global zeta integral

$$Z(s,\chi,\varphi) = \int_{\mathbb{J}} |\alpha|^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha$$

immediately takes the form of an Euler product of local zeta integrals,

$$Z(s,\chi,\varphi) = \prod_v Z_v(s,\chi_v,\varphi_v)$$

where for each place v,

$$Z_v(s,\chi_v,\varphi_v) = \int_{\mathbf{k}_v^\times} |\alpha|_v^s \chi_v(\alpha)\varphi_v(\alpha)\,\mathrm{d}_v^\times\alpha.$$

The Euler factorization will play no role in the global continuation and functional equation argument. However, we now examine some local Euler factors.

2.1. **Unramified nonarchimedean places.** Let v be a finite place of $\mathbf{k}$, and let $\mathcal{O}_v$ be the ring of integers of $\mathbf{k}_v$. With v fixed, freely drop it from the notation. Let the local character have no discretely parametrized part,

$$\chi : \mathbf{k}^\times \longrightarrow \mathbb{C}^\times, \quad \chi(\alpha) = |\alpha|^{s_o}$$

(for any global Hecke character, this holds at almost all the finite places), and let the local Schwartz function be the characteristic function of the local integers,

$$\varphi : \mathbf{k} \longrightarrow \mathbb{C}, \quad \varphi(\alpha) = 1_{\mathcal{O}}(\alpha).$$

To compute the local integral

$$Z(s) = \int_{\mathbf{k}^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha,$$

take a uniformizer (valuation-1 element) ϖ. Thus $|\varpi| = \mathrm{N}\mathfrak{p}^{-1}$ where $\mathfrak{p}$ is the prime ideal of $\mathcal{O}_{\mathbf{k}}$ corresponding to the place v. Normalize the multiplicative Haar measure so that $\mathcal{O}^\times$ has measure 1. Then the local integral is, allowing a slight abuse of notation at the last step,

$$Z(s) = \int_{\mathbf{k}^\times/\mathcal{O}^\times}\int_{\mathcal{O}^\times} |\alpha\eta|^{s+s_o}\varphi(\alpha\eta)\,\mathrm{d}_v^\times\eta\,\mathrm{d}^\times\alpha = \sum_{\ell=0}^{\infty} |\varpi^\ell|^{s+s_o} = (1-\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}.$$

Thus the local zeta integral gives an Euler factor of the Hecke L-function at almost all places. And the local Schwartz function is its own Fourier transform, so that at such places there is no need to replace it by its Fourier transform in the local factor of the functional equation.

The case of a ramified local character in the nonarchimedean case will be discussed later in the writeup.

2.2. **Real archimedean places.** Let $\mathbf{k}_v = \mathbb{R}$. Consider the trivial character and the Gaussian Schwartz function

$$\begin{aligned} \chi &: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times, \qquad \chi(\alpha) = 1, \\ \varphi &: \mathbb{R} \longrightarrow \mathbb{C}, \qquad \varphi(\alpha) = e^{-\pi\alpha^2}. \end{aligned}$$

The local integral is

$$Z(s) = \int_{\mathbb{R}^\times} |\alpha|^s \varphi(\alpha)\,\mathrm{d}^\times\alpha = 2\int_0^\infty t^s e^{-\pi t^2}\,\frac{\mathrm{d}t}{t} = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right).$$

This is the familiar archimedean factor from completing the basic zeta function or an even Dirichlet L-function.

On the other hand, consider the sign character (which arises as the infinite part of an odd Dirichlet character viewed as a classical Hecke character, for example) and the simplest odd Schwartz function that incorporates the Gaussian,

$$\begin{aligned} \chi &: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times, \qquad \chi(\alpha) = \mathrm{sgn}(\alpha), \\ \varphi &: \mathbb{R} \longrightarrow \mathbb{C}, \qquad \varphi(\alpha) = \alpha e^{-\pi\alpha^2}. \end{aligned}$$

Then the local integral is

$$Z(s) = \int_{\mathbb{R}^\times} |\alpha|^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha = 2\int_0^\infty t^{s+1} e^{-\pi t^2}\,\frac{\mathrm{d}t}{t} = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right).$$

This is the archimedean factor from completing an odd Dirichlet L-function.

2.3. **Complex archimedean places.** Let $\mathbf{k}_v = \mathbb{C}$. The unitary characters are

$$\chi : \mathbb{C}^\times \longrightarrow \mathbb{T}, \qquad \chi(\alpha) = \left(\frac{\alpha}{|\alpha|}\right)^m \text{ where } m \in \mathbb{Z}.$$

The Haar measure is

$$\mathrm{d}^\times\alpha = \frac{d^+\alpha}{|\alpha|^2} = \frac{r\,\mathrm{d}r\,d\theta}{r^2}.$$

Note that the change of variable $\rho = r^2$ gives $2r\,\mathrm{d}r/r^2 = d\rho/\rho$.

If $m = 0$ then let $\varphi(\alpha) = e^{-\pi|\alpha|^2}$. The local integral is

$$Z(s) = \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha = 2\pi\int_0^\infty r^{2s} e^{-\pi r^2}\,\frac{r\,\mathrm{d}r}{r^2} = \pi\cdot\pi^{-s}\Gamma(s).$$

If $m > 0$ then let $\varphi(\alpha) = \bar{\alpha}^m e^{-\pi|\alpha|^2}$. The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha = 2\pi\int_0^\infty r^{2s+m} e^{-\pi r^2}\,\frac{r\,\mathrm{d}r}{r^2} \\ &= \pi\cdot\pi^{-s-\frac{m}{2}}\Gamma\left(s+\frac{m}{2}\right). \end{aligned}$$

If $m < 0$ then let $\varphi(\alpha) = \alpha^{-m} e^{-\pi|\alpha|^2}$. The local integral is

$$\begin{aligned} Z(s) &= \int_{\mathbb{C}^\times} |\alpha|_{\mathbb{C}}^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha = 2\pi\int_0^\infty r^{2s-m} e^{-\pi r^2}\,\frac{r\,\mathrm{d}r}{r^2} \\ &= \pi\cdot\pi^{-s+\frac{m}{2}}\Gamma\left(s-\frac{m}{2}\right). \end{aligned}$$

Thus in all cases,

$$Z(s) = \pi\cdot\pi^{-s-\frac{|m|}{2}}\Gamma\left(s+\frac{|m|}{2}\right).$$

A more classical normalization is to multiply the measure by $2/\pi$ and to replace $e^{-\pi|\alpha|^2}$ by $e^{-2\pi|\alpha|^2}$ in the Schwartz functions. Then in particular when $m=0$, the local integral is $2\cdot(2\pi)^{-s}\Gamma(s)$, and by Legendre's duplication formula

$$\Gamma(z)\Gamma(z+1/2)=2^{1-2z}\pi^{1/2}\Gamma(2z)$$

this is the product of the even and odd real archimedean factors from a moment ago. The classical normalization fits tidily with the factorization of the Dedekind zeta function of a CM-extension.

Part 2. Global Theory

This part of the writeup establishes the global continuation and functional equation for the adelic zeta integral. The method is laid out to look similar to Riemann's original argument.

3. Convergence

This section shows that the half-zeta integral

$$\int_{|\alpha|\geq 1}|\alpha|^s\chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha$$

converges for all $s\in\mathbb{C}$. We may take the Schwartz function to be monomial, $\varphi=\bigotimes_v\varphi_v$, since any Schwartz function is a finite sum of such. The Schwartz function decays rapidly,

$$|\varphi(\alpha)|\leq C_{\varphi,n}\prod_v\sup\{|\alpha_v|_v,1\}^{-n},\quad \alpha\in\mathbb{A},\ n\in\mathbb{Z}^+.$$

(The product is nontrivial at only finitely many terms, and similarly for products to come in this section except the last one.) Since $\sup\{r,1\}=r^{1/2}\sup\{r^{1/2},r^{-1/2}\}$ for any $r>0$, the bounds become (omitting constants from now on)

$$\begin{aligned}|\varphi(\alpha)|&\lesssim|\alpha|^{-n/2}\prod_v\sup\{|\alpha_v|_v^{1/2},|\alpha_v|_v^{-1/2}\}^{-n}\\&=|\alpha|^{-n/2}\prod_v\inf\{|\alpha_v|_v^{n/2},|\alpha_v|_v^{-n/2}\},\quad \alpha\in\mathbb{J},\ n\in\mathbb{Z}^+,\end{aligned}$$

and so furthermore

$$|\varphi(\alpha)|\lesssim\prod_v\inf\{|\alpha_v|_v^{n/2},|\alpha_v|_v^{-n/2}\},\quad |\alpha|\geq 1,\ n\in\mathbb{Z}^+.$$

We may assume that χ is discretely parametrized by absorbing its continuous parameter into s. Thus, letting $\sigma=\mathrm{Re}(s)$,

$$\begin{aligned}\left|\int_{|\alpha|\geq 1}|\alpha|^s\chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha\right|&\lesssim\int_{|\alpha|\geq 1}|\alpha|^\sigma\prod_v\inf\{|\alpha_v|_v^{n/2},|\alpha_v|_v^{-n/2}\}\mathrm{d}^\times\alpha\\&\leq\prod_v\int_{\mathbf{k}_v^\times}|\alpha_v|^\sigma\inf\{|\alpha_v|_v^{n/2},|\alpha_v|_v^{-n/2}\}\,\mathrm{d}^\times\alpha,\quad n\in\mathbb{Z}^+.\end{aligned}$$

Each nonarchimedean integral is a sum over $\mathbb{Z}$ that converges absolutely in both directions for $n>2|\sigma|$. Specifically, it is

$$\sum_{e\geq 0}q_v^{-e(\sigma+n/2)}+\sum_{e>0}q_v^{e(\sigma-n/2)}=\frac{1-q_v^{-n}}{(1-q_v^{-\sigma-n/2})(1-q_v^{\sigma-n/2})}\,.$$

The archimedean integrals are similar. Thus for any s the product is dominated by the convergent product for $\zeta_{\mathbf{k}}(\sigma + n/2)\zeta_{\mathbf{k}}(-\sigma + n/2)/\zeta_{\mathbf{k}}(n)$ for all sufficiently large n.

4. Elements of Adelic Harmonic Analysis

Let $\mathbb{T}$ denote the circle group of complex numbers of absolute value 1. Take an *adelic additive unitary character*

$$\psi : \mathbb{A} \longrightarrow \mathbb{T}$$

that is trivial on $\mathbf{k}$,

$$\psi(\mathbf{k}) = 1.$$

The *adelic Fourier transform* of a Schwartz function $\varphi : \mathbb{A} \longrightarrow \mathbb{C}$ is

$$\mathcal{F}\varphi : \mathbb{A} \longrightarrow \mathbb{C}, \qquad (\mathcal{F}\varphi)(x) = \int_{\mathbb{A}} \varphi(y)\psi(-xy)\,\mathrm{d}y.$$

Here the Haar measure on $\mathbb{A}$ is normalized so that $(\mathcal{F}\mathcal{F}\varphi)(x) = \varphi(-x)$.

We construct such a character, if only to show that one exists and perhaps to give the reader some grounding. First define an additive unitary character for each rational prime. For a finite prime p of $\mathbb{Q}$ the character is

$$e_p(x) = \exp(-2\pi i x), \quad x \in \mathbb{Q}_p,$$

whose meaning and continuity are clear on the dense subset $\mathbb{Z}[1/p]$ of $\mathbb{Q}_p$ and therefore on all of $\mathbb{Q}_p$. For the infinite prime of $\mathbb{Q}$ the character is

$$e_\infty(x) = \exp(2\pi i x), \quad x \in \mathbb{R}.$$

Thus $\bigotimes_p e_p(\mathbb{Q}) = 1$, the product including the infinite prime. Now return to the general number field $\mathbf{k}$, and take each local character ψ_v to be the trace followed by the corresponding character,

$$\psi_v = e_p \circ \operatorname{tr}_{\mathbf{k}_v/\mathbb{Q}_p}, \quad v \mid p.$$

(Thus the kernel of ψ_v is the local inverse different, simply the local integers away from ramification.) The product of the local characters,

$$\psi = \bigotimes_v \psi_v : \mathbb{A} \longrightarrow \mathbb{T},$$

is an adelic additive unitary character. Not only does the general adelic additive unitary character decompose as a product of local characters in this fashion, but in fact we will later see that the general adelic additive unitary character can be normalized to this particular ψ.

For any Schwartz function φ and any idele α, define the dilation

$$\varphi_\alpha : \mathbb{A} \longrightarrow \mathbb{C}, \qquad \varphi_\alpha(x) = \varphi(\alpha x).$$

The relation between the Fourier transforms of the dilation and of the original function is

$$\mathcal{F}(\varphi_\alpha) = |\alpha|_{\mathbb{A}}^{-1}(\mathcal{F}\varphi)_{\alpha^{-1}}.$$

To see this, compute

$$\begin{aligned}(\mathcal{F}(\varphi_\alpha))(x) &= \int_{\mathbb{A}} \varphi_\alpha(y)\psi(-xy)\,\mathrm{d}y = \int_{\mathbb{A}} \varphi(\alpha y)\psi(-\alpha^{-1}x\alpha y)\,d(\alpha y)/|\alpha|_{\mathbb{A}} \\ &= |\alpha|_{\mathbb{A}}^{-1}(\mathcal{F}\varphi)(\alpha^{-1}x).\end{aligned}$$

The *adelic Poisson summation formula* is derived as follows. Given a Schwartz function $f : \mathbb{A} \longrightarrow \mathbb{C}$, symmetrize to obtain a $\mathbf{k}$-periodic function

$$F : \mathbb{A} \longrightarrow \mathbb{C}, \qquad F(x) = \sum_{k \in \mathbf{k}} f(x+k).$$

The symmetrized function is equal to its Fourier series,

$$F(x) = \sum_{k \in \mathbf{k}} c_k(F)\psi(kx),$$

where the kth Fourier coefficient is

$$\begin{aligned} c_k(F) &= \int_{\mathbb{A}/\mathbf{k}} F(y)\psi(-ky)\,\mathrm{d}y \\ &= \int_{\mathbb{A}/\mathbf{k}} \sum_{\kappa \in \mathbf{k}} f(y+\kappa)\psi(-k(y+\kappa))\,\mathrm{d}y \\ &= \int_{\mathbb{A}} f(y)\psi(-ky)\,\mathrm{d}y \qquad \text{since } \psi \text{ is additive and } \psi(\mathbf{k}) = 1 \\ &= (\mathcal{F}f)(k). \end{aligned}$$

This gives the Poisson summation formula,

$$\sum_{k \in \mathbf{k}} f(x+k) = \sum_{k \in \mathbf{k}} (\mathcal{F}f)(k)\psi(kx),$$

and especially when $x = 0$ we have

$$\sum_{k \in \mathbf{k}} f(k) = \sum_{k \in \mathbf{k}} (\mathcal{F}f)(k).$$

Let A be a topological group (locally compact and countably-based) and B a closed subgroup. The *mock-Fubini formula* is

$$\int_A f(\alpha)\,\mathrm{d}\alpha = \int_{A/B} F(\overline{\alpha})\,\mathrm{d}\overline{\alpha} \qquad \text{where} \quad F(\overline{\alpha}) = \int_B f(\alpha\beta)\,\mathrm{d}\beta.$$

5. Continuation and Functional Equation

Again, for a Hecke character $\chi : \mathbb{J} \longrightarrow \mathbb{C}^\times$ and a Schwartz function $\varphi : \mathbb{A} \longrightarrow \mathbb{C}$, the associated adelic zeta integral is

$$Z(s, \chi, \varphi) = \int_{\mathbb{J}} |\alpha|^s \chi(\alpha)\varphi(\alpha)\,\mathrm{d}^\times\alpha,$$

convergent for s in a right half-plane.

To establish its analytic continuation and functional equation, begin by defining an adelic theta function,

$$\theta_\varphi(\alpha) = \sum_{k \in \mathbf{k}} \varphi(\alpha k), \quad \alpha \in \mathbb{J}.$$

The adelic Poisson summation formula and then the formula for the Fourier transform of a dilation give the transformation law for the theta function,

$$\theta_\varphi(\alpha) = \sum_{k \in \mathbf{k}} \varphi_\alpha(k) = \sum_{k \in \mathbf{k}} (\mathcal{F}(\varphi_\alpha))(k) = |\alpha|^{-1} \sum_{k \in \mathbf{k}} (\mathcal{F}\varphi)_{\alpha^{-1}}(k) = |\alpha|^{-1}\theta_{\mathcal{F}\varphi}\left(\alpha^{-1}\right).$$

A calculation that uses the mock-Fubini formula at the first step shows that the theta function contributes to the zeta integral,

$$\begin{aligned} Z(s,\chi,\varphi) &= \int_{\mathbb{J}/\mathbf{k}^\times} \sum_{k\in\mathbf{k}^\times} |\alpha k|^s \chi(\alpha k)\varphi(\alpha k)\,\mathrm{d}^\times\alpha \\ &= \int_{\mathbb{J}/\mathbf{k}^\times} |\alpha|^s\chi(\alpha)\left(\sum_{k\in\mathbf{k}}\varphi(\alpha k)-\varphi(0)\right)\mathrm{d}^\times\alpha \\ &= \int_{\mathbb{J}/\mathbf{k}^\times} |\alpha|^s\chi(\alpha)\big(\theta_\varphi(\alpha)-\varphi(0)\big)\,\mathrm{d}^\times\alpha. \end{aligned}$$

The zeta integral splits into two terms. Let

$$S=\{\alpha\in\mathbb{J}:|\alpha|\le 1\}/\mathbf{k}^\times,\qquad T=\{\alpha\in\mathbb{J}:|\alpha|\ge 1\}/\mathbf{k}^\times.$$

Then $S\cup T=\mathbb{J}/\mathbf{k}^\times$ and $S\cap T$ has measure 0, so that

$$\begin{aligned} Z(s,\chi,\varphi) = \int_S |\alpha|^s\chi(\alpha)\big(\theta_\varphi(\alpha)-\varphi(0)\big)\,\mathrm{d}^\times\alpha \\ + \int_T |\alpha|^s\chi(\alpha)\big(\theta_\varphi(\alpha)-\varphi(0)\big)\,\mathrm{d}^\times\alpha. \end{aligned}$$

The transformation law for the theta function shows that the first term of the right side of the previous display is

$$\begin{aligned} &\int_S |\alpha|^s\chi(\alpha)\big(\theta_\varphi(\alpha)-\varphi(0)\big)\,\mathrm{d}^\times\alpha \\ &\qquad= \int_S |\alpha|^{s-1}\chi(\alpha)\big(\theta_{\mathcal{F}\varphi}(\alpha^{-1})-(\mathcal{F}\varphi)(0)\big)\,\mathrm{d}^\times\alpha \\ &\qquad\quad - \varphi(0)\int_S |\alpha|^s\chi(\alpha)\,\mathrm{d}^\times\alpha \\ &\qquad\quad + (\mathcal{F}\varphi)(0)\int_S |\alpha|^{s-1}\chi(\alpha)\,\mathrm{d}^\times\alpha. \end{aligned} \tag{1}$$

And substituting α^{-1} for α shows that the main term of the right side of (1) is

$$\begin{aligned} &\int_S |\alpha|^{s-1}\chi(\alpha)\big(\theta_{\mathcal{F}\varphi}(\alpha^{-1})-(\mathcal{F}\varphi)(0)\big)\,\mathrm{d}^\times\alpha \\ &\qquad= \int_T |\alpha|^{1-s}\chi^{-1}(\alpha)\big(\theta_{\mathcal{F}\varphi}(\alpha)-(\mathcal{F}\varphi)(0)\big)\,\mathrm{d}^\times\alpha. \end{aligned}$$

To study the other two terms of the right side of (1), again use the mock-Fubini formula

$$\int_A f(\alpha)\,\mathrm{d}\alpha = \int_{A/B} F(\overline{\alpha})\,\mathrm{d}\overline{\alpha}\qquad\text{where}\quad F(\overline{\alpha})=\int_B f(\alpha\beta)\,\mathrm{d}\beta.$$

Let $A=S$, $B=\mathbb{J}^1/\mathbf{k}^\times$, and $f(\alpha)=|\alpha|^s\chi(\alpha)$. Then the inner integral is

$$\begin{aligned} F(\overline{\alpha}) &= \int_{\mathbb{J}^1/\mathbf{k}^\times} |\alpha\beta|^s\chi(\alpha\beta)\,\mathrm{d}^\times\beta = |\alpha|^s\chi(\alpha)\int_{\mathbb{J}^1/\mathbf{k}^\times}\chi(\beta)\,\mathrm{d}^\times\beta \\ &= \begin{cases} |\overline{\alpha}|^s\mathrm{vol}(\mathbb{J}^1/\mathbf{k}^\times) & \text{if } \chi=1, \\ 0 & \text{if } \chi\ne 1. \end{cases} \end{aligned}$$

Here we are using the fact that B is a compact group, although A isn't a group at all. And since $A/B \cong (0,1]$, where the map is $\overline{\alpha} \mapsto |\overline{\alpha}|$, the nontrivial case of the outer integral is (omitting constants for the moment)

$$\int_{S/(\mathbb{J}^1/\mathbf{k}^\times)} |\overline{\alpha}|^s \, d^\times \overline{\alpha} = \int_0^1 t^s \frac{\mathrm{d}t}{t} = \frac{1}{s}.$$

Thus the second term of the right side of (1) is

$$-\varphi(0) \int_S |\alpha|^s \chi(\alpha) \,\mathrm{d}^\times \alpha = \begin{cases} -\mathrm{vol}(\mathbb{J}^1/\mathbf{k}^\times)\,\dfrac{\varphi(0)}{s} & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1. \end{cases}$$

Similarly the third term of the right side of (1) is

$$(\mathcal{F}\varphi)(0) \int_S |\alpha|^{s-1} \chi(\alpha) \,\mathrm{d}^\times \alpha = \begin{cases} -\mathrm{vol}(\mathbb{J}^1/\mathbf{k}^\times)\,\dfrac{(\mathcal{F}\varphi)(0)}{1-s} & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1. \end{cases}$$

This analysis gives the desired properties of the zeta integral. If $\chi = 1$ then

$$\begin{aligned} Z(s,1,\varphi) = \int_T \Big(|\alpha|^s \big(\theta_\varphi(\alpha) - \varphi(0)\big) + |\alpha|^{1-s} \big(\theta_{\mathcal{F}\varphi}(\alpha) - (\mathcal{F}\varphi)(0)\big) \Big) \,\mathrm{d}^\times \alpha \\ - \mathrm{vol}(\mathbb{J}^1/\mathbf{k}^\times) \left(\frac{\varphi(0)}{s} + \frac{(\mathcal{F}\varphi)(0)}{1-s} \right), \end{aligned}$$

while if $\chi \neq 1$ then the two terms that contribute poles vanish, and so

$$Z(s,\chi,\varphi) = \int_T \Big(|\alpha|^s \chi(\alpha) \big(\theta_\varphi(\alpha) - \varphi(0)\big) + |\alpha|^{1-s} \chi^{-1}(\alpha) \big(\theta_{\mathcal{F}\varphi}(\alpha) - (\mathcal{F}\varphi)(0)\big) \Big) \,\mathrm{d}^\times \alpha.$$

In both cases, the integral is an entire function of s, as discussed earlier. In either case, the zeta integral $Z(s,\chi,\varphi)$ is visibly invariant under the substitution $(s,\chi,\varphi) \mapsto (1-s,\chi^{-1},\mathcal{F}\varphi)$. When $\chi = 1$, it is meromorphic with simple poles at $s = 0$ and $s = 1$, and when $\chi \neq 1$, it is entire. That is, we have established that

- $Z(s,\chi,\varphi)$ has a continuation to the complex plane, and the continuation satisfies the functional equation
$$Z(s,\chi,\varphi) = Z(1-s,\chi^{-1},\mathcal{F}\varphi).$$
- The continuation of $Z(s,\chi,\varphi)$ is analytic when $\chi \neq 1$, and it is meromorphic with simple poles at $s = 0$ and $s = 1$ when $\chi = 1$.

Later in this writeup we will view the zeta integral as an (s,χ)-parametrized function $Z(s,\chi)$ of Schwartz functions φ, i.e., as a distribution. From this point of view, the functional equation is

$$Z(s,\chi) = \mathcal{F}Z(1-s,\chi^{-1}).$$

This is the version of the functional equation that will be quoted at the very end of the writeup.

Part 3. Local Theory

Each local factor of the zeta integral can be continued as well. Doing so leads naturally to an L-function and an ε-factor associated to the Hecke character χ. Eventually, combining the local results with the global work produces a continuation and a function equation for the L-function.

6. The Local Zeta Integral

We work over a local field F. Thus F is nonarchimedean, or $F = \mathbb{R}$, or $F = \mathbb{C}$. Let $\mathcal{S}(F)$ denote the space of Schwartz functions on F. With F clear in our minds we freely suppress it from the notation, for example writing $\mathcal{S}$ rather than $\mathcal{S}(F)$.

Let

$$\chi : F^\times \longrightarrow \mathbb{C}^\times$$

be a character, not necessarily unitary. For any Schwartz function $\varphi \in \mathcal{S}$ and for a complex parameter $s \in \mathbb{C}$, the local zeta integral is formally (not yet discussing convergence)

$$\boxed{Z(s,\chi,\varphi) = \int_{F^\times} \varphi(x)\chi(x)|x|^s \, \mathrm{d}^\times x.}$$

In the archimedean case, the multiplicative Haar measure in the definition is

$$\mathrm{d}^\times x = \mathrm{d}x/|x| \text{ where } \begin{cases} \mathrm{d}x \text{ is Lebesgue measure and } |x| = \sqrt{x^2} & \text{if } F = \mathbb{R}, \\ \mathrm{d}x \text{ is twice Lebesgue measure and } |x| = x\overline{x} & \text{if } F = \mathbb{C}. \end{cases}$$

In the nonarchimedean case the multiplicative Haar measure is determined up to constant multiple, and soon we will normalize it.

7. Spaces of Distributions

Introduce two spaces:

- The Schwartz functions that vanish to all orders at 0,
$$\mathcal{S}_0 = \{\varphi \in \mathcal{S} : \varphi^{(n)}(0) = 0 \text{ for all } n \geq 0\}.$$
- The Taylor expansions of Schwartz functions at 0,
$$\mathcal{T} = \{\sum_{n\geq 0} (\varphi^{(n)}(0)/n!)x^n : \varphi \in \mathcal{S}\}.$$

(If F is nonarchimedean then $\mathcal{S}_0$ is simply the Schwartz functions that vanish at 0, and $\mathcal{T} = \{\varphi(0) : \varphi \in \mathcal{S}\}$ is simply a copy of $\mathbb{C}$.) Then we have a short exact sequence

$$0 \longrightarrow \mathcal{S}_0 \xrightarrow{\text{inc}} \mathcal{S} \longrightarrow \mathcal{T} \longrightarrow 0.$$

Introduce two more spaces:

- The tempered distributions that are supported at 0,
$$\mathcal{S}'_{\{0\}} = \{u \in \mathcal{S}' : \operatorname{supp}(u) \subset \{0\}\}.$$
These are the finite linear combinations δ-derivatives, counting δ as its own zeroth derivative. Note that $\mathcal{S}'_{\{0\}} = \mathcal{T}'$.
- The restrictions of tempered distributions to $\mathcal{S}_0$,
$$\mathcal{S}'\big|_{\mathcal{S}_0} = \{u\big|_{\mathcal{S}_0} : u \in \mathcal{S}'\}.$$
By the Hahn–Banach Theorem, $\mathcal{S}'\big|_{\mathcal{S}_0}$ is all of $\mathcal{S}'_0$.

The previous short exact sequence dualizes to another,

$$0 \longrightarrow \mathcal{S}'_{\{0\}} \longrightarrow \mathcal{S}' \xrightarrow{\text{res}} \mathcal{S}'\big|_{\mathcal{S}_0} \longrightarrow 0.$$

For any $a \in F^\times$, the right-translation operator R_a on Schwartz functions $\varphi \in \mathcal{S}$ is defined by the condition

$$(R_a\varphi)(x) = \varphi(xa).$$

(Since F is a field, right-translation and left-translation are the same, but the rule $R_{ab} = R_a \circ R_b$ holds with no reference to commutativity, whereas the general rule for left-translation is $L_{ab} = L_b \circ L_a$.) The right-translation operator R_a on tempered distributions $u \in \mathcal{S}'$ is defined as an adjoint,

$$\langle R_a u\,,\, \varphi\rangle = \langle u\,,\, R_{a^{-1}}\varphi\rangle.$$

(The inverse ensures that again the rule $R_{ab} = R_a \circ R_b$ holds with no reference to commutativity.) A tempered distribution $u \in \mathcal{S}'$ is an *eigendistribution* if for some character $\chi : F^\times \longrightarrow \mathbb{C}^\times$, not necessarily unitary,

$$R_a u = \chi(a)u \quad \text{for all } a \in F^\times.$$

The eigendistributions having one particular character χ form the *eigenspace* $\mathcal{S}'(\chi)$.

Let $\chi : F^\times \longrightarrow \mathbb{C}^\times$ be a character, not necessarily unitary. The previous short exact sequence of distributions restricts to eigenspaces,

$$0 \longrightarrow \mathcal{S}'_{\{0\}}(\chi) \longrightarrow \mathcal{S}'(\chi) \xrightarrow{\text{res}} \mathcal{S}'\big|_{\mathcal{S}_0}(\chi),$$

where now the restriction map need not surject: a χ-eigendistribution on $\mathcal{S}_0$ lifts to a distribution on $\mathcal{S}$, but the lift need not again be a χ-eigendistribution.

The first object of the previous sequence is at most one-dimensional. Specifically (letting *na* stand for *nonarchimedean*),

$$\mathcal{S}'_{\{0\}} = \begin{cases} \mathbb{C}\delta = \mathcal{S}'_{\{0\}}(\chi_0) & & \text{if } F = \text{na}, \\ \bigoplus_{k\ge 0} \mathbb{C}D^k\delta = \bigoplus_{k\ge 0} \mathcal{S}'_{\{0\}}(\chi_k) & \text{where } \chi_k(x) = x^{-k} & \text{if } F = \mathbb{R}, \\ \bigoplus_{k,\ell\ge 0} \mathbb{C}D^{k,\ell}\delta = \bigoplus_{k,\ell\ge 0} \mathcal{S}'_{\{0\}}(\chi_{k,\ell}) & \text{where } \chi_{k,\ell}(x) = x^{-k}\overline{x}^{-\ell} & \text{if } F = \mathbb{C}. \end{cases}$$

We quote that the third object of the previous sequence is exactly one-dimensional. Specifically, if we let λ_χ denote the distribution that integrates against χ over $F^\times$,

$$\lambda_\chi : \varphi \longmapsto \int_{F^\times} \varphi(x)\chi(x)\,\mathrm{d}^\times x,$$

then

$$\mathcal{S}'\big|_{\mathcal{S}_0}(\chi) = \mathbb{C}\lambda_\chi.$$

Thus the sequence takes various possible forms:

$$\begin{aligned} F = \text{na},\ \chi \ne \chi_o :&\quad 0 \longrightarrow 0 \longrightarrow \mathcal{S}'(\chi) \longrightarrow \mathbb{C}\lambda_\chi, \\ F = \mathbb{R},\ \chi \ne \chi_k :&\quad 0 \longrightarrow 0 \longrightarrow \mathcal{S}'(\chi) \longrightarrow \mathbb{C}\lambda_\chi, \\ F = \mathbb{C},\ \chi \ne \chi_{k,\ell} :&\quad 0 \longrightarrow 0 \longrightarrow \mathcal{S}'(\chi) \longrightarrow \mathbb{C}\lambda_\chi, \\ F = \text{na},\ \chi = \chi_o :&\quad 0 \longrightarrow \mathbb{C}\delta \longrightarrow \mathcal{S}'(\chi_o) \longrightarrow \mathbb{C}\lambda_{\lambda_{\chi_o}}, \\ F = \mathbb{R},\ \chi = \chi_k :&\quad 0 \longrightarrow \mathbb{C}D^k\delta \longrightarrow \mathcal{S}'(\chi_k) \longrightarrow \mathbb{C}\lambda_{\chi_k}, \\ F = \mathbb{C},\ \chi = \chi_{k,\ell} :&\quad 0 \longrightarrow \mathbb{C}D^{k,\ell}\delta \longrightarrow \mathcal{S}'(\chi_{k,\ell}) \longrightarrow \mathbb{C}\lambda_{\chi_{k,\ell}}. \end{aligned}$$

In the first three cases $\dim(\mathcal{S}'(\chi)) \le 1$, and in the last three $\dim(\mathcal{S}'(\chi)) \le 2$. The result that we want is:

$$\textit{In all cases, } \dim(\mathcal{S}'(\chi)) = 1.$$

That is:

- In the first three cases, where $\mathcal{S}'(\chi)$ injects into $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi)$, we need to show that it surjects by showing that a lift of λ_χ from $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi)$ to $\mathcal{S}'$ lies in $\mathcal{S}'(\chi)$.
- In the last three cases, where the map from $\mathcal{S}'(\chi)$ to $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi)$ has a one-dimensional kernel, we need to show that its image is trivial by showing that *no* lift of λ_χ from $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi)$ to $\mathcal{S}'$ lies in $\mathcal{S}'(\chi)$.

In sections 7.1–7.3 we will establish the result in the nonarchimedean case modulo the invocation that the right-hand link of the sequence is one-dimensional, meeting some fruitful ideas in the process. Then, introducing Fourier analysis in section 8.1 will give two nonzero basis elements of $\mathcal{S}'(\chi)$ in section 8.2, so that the two must be proportional. The constant of proportionality is the *local ε-factor*, and we will compute it in sections 8.3–8.4.

7.1. **The nonarchimedean unramified case.** Now assume that F is nonarchimedean. Let

$$\begin{aligned}
\mathcal{O} &= \text{the ring of integers of } F,\\
\mathcal{P} &= \text{the maximal ideal of } \mathcal{O},\\
\varpi &= \text{a generator of } \mathcal{P},\\
q &= |\mathcal{O}/\mathcal{P}|,\\
|u\varpi^e| &= q^{-e} \text{ for } u \in \mathcal{O}^\times \text{ and } e \in \mathbb{Z},\\
\mathrm{d}^\times x &= \text{multiplicative Haar measure, normalized so that } \int_{\mathcal{O}^\times} \mathrm{d}^\times x = 1.
\end{aligned}$$

Consider a complex parameter $s \in \mathbb{C}$ and a character $\chi : F^\times \longrightarrow \mathbb{C}^\times$. As explained above, the distribution

$$\langle \lambda\,,\, \varphi_0 \rangle = Z(s, \chi, \varphi_0)$$

spans the one-dimensional eigenspace $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi\,|\cdot|^s)$. Since each Schwartz function $\varphi_0 \in \mathcal{S}_0$ vanishes to all orders at 0, the integral is an entire function of s.

However, now consider instead a Schwartz function φ from the larger space $\mathcal{S}$ rather than from $\mathcal{S}_0$. The local zeta integral remains analytic in s for $\mathrm{Re}(s) > 0$, but for general s we can not immediately take the alleged distribution

$$\text{“}\,\langle \lambda\,,\, \varphi \rangle = Z(s, \chi, \varphi)\,\text{”}$$

as a basis element of $\mathcal{S}'(\chi\,|\cdot|^s)$, or even as an element of $\mathcal{S}'$ at all. To work around the problem, note that by the nature of nonarchimedean Schwartz functions, φ is constant on some neighborhood $\mathcal{P}^{e-1}$ of 0. Thus the related function

$$\varphi_0 = (1 - R_{\varpi^{-1}})\varphi, \quad \varphi_0(x) = \varphi(x) - \varphi(x\varpi^{-1})$$

is identically 0 on $\mathcal{P}^e$, i.e., $\varphi_0 \in \mathcal{S}_0$. Consequently, the local zeta integral of the related function,

$$Z(s, \chi, \varphi_0) = \int_{F^\times} \varphi_0(x)\chi(x)|x|^s\, \mathrm{d}^\times x = \langle \lambda\,,\, \varphi_0 \rangle$$

is entire in s. The distribution μ whose output is this related local zeta integral,

$$\langle \mu\,,\, \varphi \rangle = \langle \lambda\,,\, \varphi_0 \rangle = Z(s, \chi, \varphi_0),$$

lies in $\mathcal{S}'(\chi\,|\cdot|^s)$, because the preliminary calculation that for $a \in F^\times$ and $\varphi \in \mathcal{S}$,

$$(R_{a^{-1}}\varphi)_0 = (1 - R_{\varpi^{-1}})(R_{a^{-1}}\varphi) = R_{a^{-1}}(1 - R_{\varpi^{-1}})\varphi = R_{a^{-1}}(\varphi_0),$$

justifies the third equality in the calculation

$$\begin{aligned}\langle R_a\mu\,,\,\varphi\rangle &= \langle \mu\,,\,R_{a^{-1}}\varphi\rangle = \langle\lambda\,,\,(R_{a^{-1}}\varphi)_0\rangle = \langle\lambda\,,\,R_{a^{-1}}(\varphi_0)\rangle \\ &= \langle R_a\lambda\,,\,\varphi_0\rangle = \chi(a)|a|^s\langle\lambda\,,\,\varphi_0\rangle = \chi(a)|a|^s\langle\mu\,,\,\varphi\rangle.\end{aligned}$$

So we have proved that $\dim(\mathcal{S}'(\chi\,|\cdot|^s)) \geq 1$, provided that μ is not the zero distribution. To show that indeed μ is not the zero distribution, let $\varphi^o = 1_{\mathcal{O}}$ be the characteristic function of $\mathcal{O}$, and compute

$$\langle\mu\,,\,\varphi^o\rangle = \int_{F^\times}(1_{\mathcal{O}}(x) - 1_{\mathcal{O}}(x\varpi^{-1}))\chi(x)|x|^s\,\mathrm{d}^\times x = \int_{\mathcal{O}^\times}\chi(x)\,\mathrm{d}^\times x.$$

We are assuming that χ is unramified, and so (recalling that the multiplicative Haar measure of $\mathcal{O}^\times$ has been normalized to 1) the result is 1.

No claim is made that the basis element μ lifts the local zeta integral distribution λ from $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi\,|\cdot|^s))$ to $\mathcal{S}'(\chi\,|\cdot|^s))$. It does not, because in general $\varphi_0 \neq \varphi$ for $\varphi \in \mathcal{S}_0$. We will see that no such lift is possible for the trivial character, and so in that case μ must be a multiple of the δ distribution. To see that in fact μ is exactly δ, compute that since any given $\varphi \in \mathcal{S}$ is constant on some $\mathcal{P}^{e-1}$ we have

$$\begin{aligned}\langle\mu\,,\,\varphi\rangle &= \int_{F^\times}(\varphi(x) - \varphi(x\varpi^{-1}))\,\mathrm{d}^\times x = \int_{F^\times - \mathcal{P}^e}(\varphi(x) - \varphi(x\varpi^{-1}))\,\mathrm{d}^\times x \\ &= \int_{F^\times - \mathcal{P}^e}\varphi(x)\,\mathrm{d}^\times x - \int_{F^\times - \mathcal{P}^e}\varphi(x\varpi^{-1})\,\mathrm{d}^\times x = \int_{\mathcal{O}^\times}\varphi(\varpi^{e-1}x)\,\mathrm{d}^\times x \\ &= \varphi(0).\end{aligned}$$

That is, $\mu = \delta$ in the trivial character case as claimed.

We will continue the local zeta integral meromorphically to all $s \in \mathbb{C}$, obtaining a second basis element of $\mathcal{S}'(\chi\,|\cdot|^s)$ away from the poles of the continuation. The idea is to break the related integral $Z(s,\chi,\varphi_0)$ into two pieces and recover the original $Z(x,\chi,\varphi)$ times a factor. Compute that for $\mathrm{Re}(s) > 0$,

$$\begin{aligned}\int_{F^\times}\varphi_0(x)\chi(x)|x|^s\,\mathrm{d}^\times x &= \int_{F^\times}\varphi(x)\chi(x)|x|^s\,\mathrm{d}^\times x - \int_{F^\times}\varphi(x\varpi^{-1})\chi(x)|x|^s\,\mathrm{d}^\times x \\ &= (1 - \chi(\varpi)q^{-s})Z(s,\chi,\varphi) \\ &= L(s,\chi)^{-1}Z(s,\chi,\varphi).\end{aligned}$$

In terms of distributions, slightly rearranging the previous display gives

$$\langle\lambda_{\chi\,|\cdot|^s}\,,\,\varphi\rangle = L(s,\chi)\langle\mu_{\chi\,|\cdot|^s}\,,\,\varphi\rangle, \quad \mathrm{Re}(s) > 0.$$

However, the right side of the previous display is the product of

- an L-factor that is meromorphic in s,
- and a local zeta integral that is entire in s.

Thus we use the right side to continue the left side meromorphically in s,

$$\langle\lambda_{\chi\,|\cdot|^s}\,,\,\varphi\rangle = L(s,\chi)\langle\mu_{\chi\,|\cdot|^s}\,,\,\varphi\rangle, \quad s \in \mathbb{C}.$$

And so $L(s,\chi)$ is naturally a ratio of two basis-elements of a one-dimensional distribution space, away from poles.

7.2. **The nonarchimedean ramified case.** If χ is ramified then for any integers m and n with $m \geq n$,

$$\int_{\mathcal{P}^n - \mathcal{P}^m} \chi(x)|x|^s \, \mathrm{d}^\times x = \sum_{e=n}^{m-1} \chi(\varpi)^e q^{-es} \int_{\mathcal{O}^\times} \chi(x) \, \mathrm{d}^\times x = 0.$$

It follows that for any $\varphi \in \mathcal{S}$ (which is constant on some neighborhood of 0), the integral

$$\int_{F^\times - \mathcal{P}^n} \varphi(x)\chi(x)|x|^s \, \mathrm{d}^\times x, \quad n \gg 0$$

is independent of n once n is large enough. That is, the improper integral

$$Z(s, \chi, \varphi) = \int_{F^\times} \varphi(x)\chi(x)|x|^s \, \mathrm{d}^\times x$$

converges. Also, if we specify $\varphi^o = 1_{\mathcal{O}^\times} \cdot \chi^{-1}$ then

$$Z(s, \chi, \varphi^o) = \int_{\mathcal{O}^\times} \mathrm{d}^\times x = 1,$$

Thus the distribution

$$\lambda \in \mathcal{S}'\big|_{\mathcal{S}_0}, \quad \langle \lambda \,,\, \varphi \rangle = Z(s, \chi, \varphi)$$

is nonzero and immediately extends from $\mathcal{S}'\big|_{\mathcal{S}_0}(\chi\,|\cdot|^s)$ to $\mathcal{S}'(\chi\,|\cdot|^s)$. The natural definition of the local L-factor in the ramified case is therefore

$$L(s, \chi) = 1.$$

We have now shown that $\dim(\mathcal{S}'(\chi)) \geq 1$ for all χ in the nonarchimedean case, unramified or ramified, and so $\dim(\mathcal{S}'(\chi)) = 1$ for all nontrivial χ in the nonarchimedean case.

7.3. **The nonarchimedean trivial case.** In the nonarchimedean case, we still need to prove that $\mathcal{S}'(\chi_o)$ is only one-dimensional. Let φ denote a generic Schwartz function. Recall the first short exact sequence from earlier, now augmented by a left inverse of its inclusion map,

$$0 \longrightarrow \mathcal{S}_0 \underset{1 - *(0)1_{\mathcal{O}}}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathcal{S} \longrightarrow \mathcal{T} \longrightarrow 0.$$

Here $1 - *(0)1_{\mathcal{O}}$ denotes the map from $\mathcal{S}$ to $\mathcal{S}_0$ that takes any φ to $\varphi - \varphi(0)1_{\mathcal{O}}$, and so $(1 - *(0)1_{\mathcal{O}}) \circ \mathrm{inc} = 1$ on $\mathcal{S}_0$. As before, dualize and quote the Hahn–Banach Theorem to get a second short exact sequence, now augmented by a right inverse of its restriction map,

$$0 \longrightarrow \mathcal{S}'_{\{0\}} \longrightarrow \mathcal{S}' \underset{(1 - *(0)1_{\mathcal{O}})^*}{\overset{\mathrm{res}}{\rightleftarrows}} \mathcal{S}'\big|_{\mathcal{S}'_0} \longrightarrow 0$$

where $\mathrm{res} \circ (1 - *(0)1_{\mathcal{O}})^* = 1$ on $\mathcal{S}'\big|_{\mathcal{S}_0}$.

Let $\lambda = \lambda_{\chi_o} \in \mathcal{S}'\big|_{\mathcal{S}_0}(\chi_o)$ be the basic integration distribution on $\mathcal{S}_0$,

$$\langle \lambda \,,\, \varphi_0 \rangle = \int_{F^\times} \varphi_0(x) \, \mathrm{d}^\times x, \quad \varphi_0 \in \mathcal{S}_0,$$

and let $\tilde{\lambda} \in \mathcal{S}'$ denote the corresponding lift from the sequence,

$$\tilde{\lambda} = (1 - *(0)1_{\mathcal{O}})^* \lambda \in \mathcal{S}'.$$

That is to say,

$$\langle\tilde{\lambda}\,,\,\varphi\rangle = \langle\lambda\,,\,\varphi - \varphi(0)1_{\mathcal{O}}\rangle, \quad \varphi \in \mathcal{S}.$$

The general lift of λ to $\mathcal{S}'$ is $\tilde{\lambda} + c\delta$ where $c \in \mathbb{C}$. We want to show that:

No lift of λ to $\mathcal{S}'$ is $F^\times$-invariant.

But δ is $F^\times$-invariant, and so it suffices to show that the particular lift $\tilde{\lambda}$ is not $F^\times$-invariant. Compute that for any $a \in F^\times$ and any $\varphi \in \mathcal{S}'$,

$$\langle R_a\tilde{\lambda}\,,\,\varphi\rangle = \langle\tilde{\lambda}\,,\,R_{a^{-1}}\varphi\rangle = \langle\lambda\,,\,R_{a^{-1}}\varphi - (R_{a^{-1}}\varphi)(0)1_{\mathcal{O}}\rangle = \langle\lambda\,,\,R_{a^{-1}}\varphi - \varphi(0)1_{\mathcal{O}}\rangle$$

(showing by a small exercise that $R_a\tilde{\lambda}$ is again a lift of λ), and in particular for $a = 1$,

$$\langle\tilde{\lambda}\,,\,\varphi\rangle = \langle\lambda\,,\,\varphi - \varphi(0)1_{\mathcal{O}})\rangle.$$

Thus (noting that if $\varphi \in \mathcal{S}$ then $(1 - R_{a^{-1}})\varphi \in \mathcal{S}_0$)

$$\langle(1 - R_a)\tilde{\lambda}\,,\,\varphi\rangle = \langle\lambda\,,\,(1 - R_{a^{-1}})\varphi\rangle = \int_{F^\times}\big(\varphi(x) - \varphi(xa^{-1})\big)\,\mathrm{d}^\times x.$$

The integral is in general nonzero. For example, choose $\varphi = 1_{\mathcal{O}}$ to get (the last equality in the display to follow is an exercise)

$$\langle(1 - R_a)\tilde{\lambda}\,,\,1_{\mathcal{O}}\rangle = \int_{F^\times}\big(1_{\mathcal{O}}(x) - 1_{\mathcal{O}}(xa^{-1})\big)\,\mathrm{d}^\times x = \mathrm{ord}(a).$$

The calculation has shown that $R_a\tilde{\lambda} = \tilde{\lambda} - \mathrm{ord}(a)\delta$. That is, $\{\tilde{\lambda}, \delta\}$ is the basis of a two-dimensional $F^\times$-representation with the action given by

$$R_a\begin{bmatrix}\tilde{\lambda}\\ \delta\end{bmatrix} = \begin{bmatrix}1 & -\mathrm{ord}(a)\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{\lambda}\\ \delta\end{bmatrix}.$$

Unlike finite-group representations, this representation is not semisimple: the subrepresentation spanned by δ does not have a complement.

The proof that $\dim(\mathcal{S}'(\chi)) = 1$ for all χ in the nonarchimedean case is complete.

8. The Local Functional Equation

8.1. **Additive characters.** Consider a particular nontrivial unitary character

$$\psi : F \longrightarrow \mathbb{T}.$$

That is,

$$\psi(x + x') = \psi(x)\psi(x'), \quad x, x' \in F,$$

and the outputs of ψ are complex values of absolute value 1. We refer to such a character simply as *additive*. One choice of the particular character is

$$\psi = \exp\circ 2\pi i\,\mathrm{Tr}_{F/\mathbb{Q}_p},$$

but this choice is not necessarily the optimal normalization; for our purposes its role is to show that there are any additive characters at all. From now on, let ψ denote some fixed additive character of F, but it need not yet be normalized in any particular way.

With ψ in place, define the *Fourier transform* on Schwartz functions,

$$\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}, \qquad (\mathcal{F}\varphi)(\xi) = \int_F \varphi(t)\psi(\xi t)\,\mathrm{d}t.$$

We understand the measure to be normalized so that it is *self-dual*, meaning that

$$\mathcal{F}\mathcal{F} = R_{-1},$$

i.e., $(\mathcal{F}\mathcal{F}\varphi)(x) = \varphi(-x)$ for all $\varphi \in \mathcal{S}$ and $x \in F$. The question of how to normalize the measure to produce the desired self-duality will be discussed soon. We remark for future reference that for any $a \in F^\times$ (exercise),

$$\mathcal{F}R_{a^{-1}} = |a|R_a\mathcal{F} \quad \text{(equality of operators on } \mathcal{S}\text{)},$$

so that consequently (exercise—the next display is not a trivial rewrite of the previous one)

$$R_a\mathcal{F} = |a|\mathcal{F}R_{a^{-1}} \quad \text{(equality of operators on } \mathcal{S}'\text{)}.$$

We invoke that any nontrivial additive character $\widetilde{\psi}$ of F is a dilation of the given one. That is, for some $a \in F^\times$,

$$\widetilde{\psi} = \psi_a \quad \text{where } \psi_a = \psi \circ a.$$

We want to choose $c \in \mathbb{R}^+$ such that the scaled measure $\tilde{d}* = c\,d*$ is self dual with respect to $\widetilde{\psi}$, granting that $d*$ is self-dual with respect to ψ. Compute first that (exercise)

$$\widetilde{\mathcal{F}}1_{\mathcal{O}} = cR_a1_{\mathcal{O}},$$

and then second that consequently (check the steps)

$$\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}1_{\mathcal{O}} = c\widetilde{\mathcal{F}}R_a1_{\mathcal{O}} = c|a|^{-1}R_{a^{-1}}\widetilde{\mathcal{F}}1_{\mathcal{O}} = c^2|a|^{-1}R_{a^{-1}}R_a1_{\mathcal{O}} = c^2|a|^{-1}1_{\mathcal{O}},$$

so that the right choice is $c = |a|^{1/2}$.

The *conductor* $\nu(\psi)$ of ψ is the negative of the exponent of the largest (as a set) $\mathcal{P}$-power where ψ is trivial,

$$\psi = 1 \text{ on } \mathcal{P}^{-\nu(\psi)} \text{ but not on } \mathcal{P}^{-(\nu(\psi)+1)}.$$

That is, the characterizing description of the conductor is

$$\psi = 1 \text{ on } \mathcal{P}^{-e} \text{ if and only if } e \leq \nu(\psi).$$

(The $\mathcal{P}$-powers form a basis of open neighborhoods of 0 in F along with being subgroups.) It is an exercise to show that if the Schwartz function φ is a function on $\mathcal{P}^n/\mathcal{P}^m$ then its Fourier transform $\mathcal{F}\varphi$ is a function on $\mathcal{P}^{-m-\nu}/\mathcal{P}^{-n-\nu}$ where $\nu = \nu(\psi)$.

Given the conductor of ψ, we use the characterizing condition to determine the conductor of ψ_a for any $a \in F^\times$. For any $e \in \mathbb{Z}$,

$$\psi_a(\mathcal{P}^{-e}) = \psi(\mathcal{P}^{-(e-\mathrm{ord}(a))}),$$

so that $\psi_a = 1$ on $\mathcal{P}^{-e}$ if and only if $\psi = 1$ on $\mathcal{P}^{-(e-\mathrm{ord}(a))}$), which in turn holds if and only if $e - \mathrm{ord}(a) \leq \nu(\psi)$, i.e., if and only if $e \leq \nu(\psi) + \mathrm{ord}(a)$. That is, by the characterizing condition,

$$\nu(\psi_a) = \nu(\psi) + \mathrm{ord}(a).$$

By choosing a suitably, we may replace our basic character by a dilation and then assume that the basic additive character has conductor 0, i.e., it is trivial on $\mathcal{O}$ but not on $\mathcal{P}^{-1}$.

Granting that the basic additive character has conductor 0, the Fourier transform of the characteristic function of the integers is

$$(\mathcal{F}1_{\mathcal{O}})(\xi) = \int_F 1_{\mathcal{O}}(t)\psi_\xi(t)\,\mathrm{d}t = \int_{\mathcal{O}} \psi_\xi(t)\,\mathrm{d}t = \begin{cases} \mu(\mathcal{O}) & \text{if } \psi \text{ is trivial on } \xi\mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Since ψ has conductor 0, the calculation shows that $\mathcal{F}1_{\mathcal{O}} = \mu(\mathcal{O})1_{\mathcal{O}}$, and so we normalize the additive measure by specifying

$$\mu(\mathcal{O}) = 1.$$

Now $\mathcal{F}1_{\mathcal{O}} = 1_{\mathcal{O}}$, and consequently $\mathcal{F}\mathcal{F}1_{\mathcal{O}} = 1_{\mathcal{O}} = R_{-1}\mathcal{O}$. For any ψ, the Fourier inversion condition that $\mathcal{F}\mathcal{F} = R_{-1}$ holds for exactly one normalization of the Haar measure. So in particular, when ψ has conductor 0, the relevant Haar measure is our normalization $\mu(\mathcal{O}) = 1$. And more generally, when ψ has conductor ν (so that $\psi = \psi^o_{\varpi^\nu}$ where ψ^o is an additive character of conductor 0), the Haar measure is normalized by the condition $\mu(\mathcal{O}) = |\varpi^\nu|^{1/2} = q^{-\nu/2}$.

8.2. **The local functional equation.** Lemma 3.6 in Kudla states that

$$\mu \in \mathcal{S}'(\chi) \implies \mathcal{F}\mu \in \mathcal{S}'(\chi^{-1}|\cdot|).$$

For the proof, recall that

$$\begin{aligned} \mathcal{F}R_{a^{-1}} &= |a|R_a\mathcal{F} \text{ (equality of operators on } \mathcal{S}), \\ R_a\mathcal{F} &= |a|\mathcal{F}R_{a^{-1}} \text{ (equality of operators on } \mathcal{S}'). \end{aligned}$$

The lemma follows from computing that for any $a \in F^\times$ and $\varphi \in \mathcal{S}$,

$$\begin{aligned} \langle R_a\mathcal{F}\mu\,,\,\varphi\rangle &= |a|\langle \mathcal{F}R_{a^{-1}}\mu\,,\,\varphi\rangle = |a|\langle R_{a^{-1}}\mu\,,\,\mathcal{F}\varphi\rangle \\ &= \chi^{-1}(a)|a|\langle\mu\,,\,\mathcal{F}\varphi\rangle = \chi^{-1}(a)|a|\langle\mathcal{F}\mu\,,\,\varphi\rangle. \end{aligned}$$

Recall that for any χ and s, the space $\mathcal{S}'(\chi|\cdot|^s)$ is spanned by $\mu_{\chi|\cdot|^s}$, where

$$\langle \mu_{\chi|\cdot|^s}\,,\,\varphi\rangle = \int_{F^\times} ((1 - R_{\varpi^{-1}})\varphi)(x)\chi(x)|x|^s\,\mathrm{d}^\times x,$$

For any χ and s, replace χ by $\chi^{-1}|\cdot|^{1-s}$ in the lemma to get that

$$\mathcal{F}\mu_{\chi^{-1}|\cdot|^{1-s}} \in \mathcal{S}'(\chi|\cdot|^s) = \mathbb{C}\mu_{\chi|\cdot|^s}.$$

That is, since the Fourier transform is an isometry and hence doesn't kill $\mu_{\chi^{-1}|\cdot|^{1-s}}$,

$$\mathcal{F}\mu_{\chi^{-1}|\cdot|^{1-s}} = \varepsilon(s,\chi,\psi)\mu_{\chi|\cdot|^s} \quad \text{for some } \varepsilon(s,\chi,\psi) \in \mathbb{C}^\times.$$

8.3. **Pre-normalizing the ε-factor calculation.** For any $s \in \mathbb{C}$, $\chi|\cdot|^s$ is ramified if and only if χ is ramified, because $|\cdot|^s = 1$ on $\mathcal{O}^\times$. Consequently the basic function φ^o from earlier, taken to 1 by μ, is the same for $\mu_{\chi|\cdot|^s}$ as it is for μ_χ. Specifically, it is

$$\varphi^o = \begin{cases} 1_{\mathcal{O}} & \text{if } \chi \text{ is unramified,} \\ 1_{\mathcal{O}^\times}\cdot\chi^{-1} & \text{if } \chi \text{ is ramified.} \end{cases}$$

Apply the relation from the end of the previous section,

$$\varepsilon(s,\chi,\psi)\mu_{\chi|\cdot|^s} = \mathcal{F}\mu_{\chi^{-1}|\cdot|^{1-s}},$$

to the basic function, giving a formula for the local ε-factor,

$$\varepsilon(s,\chi,\psi) = \langle\mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\mathcal{F}\varphi^o\rangle. \tag{2}$$

We will show that:

To compute $\varepsilon(s,\chi,\psi)$ we may assume that χ is unitary.

To see this, apply (2) twice to get

$$\begin{aligned}\varepsilon(s,\chi\,|\cdot|^t,\psi) &= \langle \mu_{\chi^{-1}|\cdot|^{-t}\,|\cdot|^{1-s}}\,,\,\mathcal{F}\varphi^o\rangle \\ &= \langle \mu_{\chi^{-1}\,|\cdot|^{1-s-t}},\chi\,,\,\mathcal{F}\varphi^o\rangle = \varepsilon(s+t,\chi,\psi).\end{aligned}$$

Thus we may assume that χ is unitary, as desired.

Next we show that:

To compute $\varepsilon(s,\chi,\psi)$ we may assume that ψ is normalized.

Fix any $a\in F^\times$, and let $\widetilde{\psi}=\psi_a$. Recall that if the measure dx is self-dual with respect to ψ then $|a|^{1/2}dx$ is self dual with respect to $\widetilde{\psi}$. Use (2) with $\widetilde{\psi}$ in place of ψ, and then at the end of the calculation use (2) again with the original ψ,

$$\begin{aligned}\varepsilon(s,\chi,\psi_a) &= \langle \mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\widetilde{\mathcal{F}}\varphi^o\rangle \\ &= |a|^{1/2}\langle \mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,R_a\mathcal{F}\varphi^o\rangle \\ &= |a|^{1/2}\langle R_{a^{-1}}\mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\mathcal{F}\varphi^o\rangle \\ &= |a|^{1/2}|a|^{s-1}\chi(a)\langle \mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\mathcal{F}\varphi^o\rangle \\ &= |a|^{s-1/2}\chi(a)\varepsilon(s,\chi,\psi).\end{aligned}$$

Thus we may assume that ψ is normalized, as desired.

8.4. **The ε-factor calculation.** From the previous section, we have the formula for $\varepsilon(s,\chi,\psi)$

$$\varepsilon(s,\chi,\psi) = \langle \mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\mathcal{F}\varphi^o\rangle \tag{3}$$

and the reduction identity

$$\varepsilon(s,\chi,\psi_a) = |a|^{s-1/2}\chi(a)\varepsilon(s,\chi,\psi).$$

If χ is unramified then first take ψ normalized to have conductor 0. As discussed, we then have $\mathcal{F}1_{\mathcal{O}}=1_{\mathcal{O}}$, and so (3) becomes

$$\varepsilon(s,\chi,\psi) = \langle \mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,1_{\mathcal{O}}\rangle = 1.$$

Combine the reduction identity with the previous display to obtain for any $a\in F^\times$,

$$\varepsilon(s,\chi,\psi_a) = |a|^{s-1/2}\chi(a).$$

The additive character ψ_a has conductor $\nu=\mathrm{ord}(a)$, so rewrite the previous display, now letting ψ denote a general additive character of conductor ν,

$$\boxed{\varepsilon(s,\chi,\psi) = q^{(1/2-s)\nu}\chi(\varpi^\nu),\quad F=\text{na},\ \chi\text{ unramified.}}$$

If χ is ramified then its conductor $c=c(\chi)\in\mathbb{Z}^+$ is defined as the exponent of the largest (as a set) $\mathcal{P}$-power such that χ is trivial on that $\mathcal{P}$-power translate about 1,

$$\chi = 1 \text{ on } 1+\mathcal{P}^{c(\chi)} \text{ but not on } 1+\mathcal{P}^{c(\chi)-1}.$$

(The $\mathcal{P}$-power translates about 1 form a basis of open neighborhoods of 1 in $F^\times$ along with being subgroups.)

Let $c\in\mathbb{Z}^+$ denote the conductor of χ, and assume first that ψ has conductor 0. The basic function $\varphi^o = 1_{\mathcal{O}^\times}\cdot\chi^{-1}$ is a function on $\mathcal{O}/\mathcal{P}^c$, and so its Fourier transform $\mathcal{F}\varphi^o$ is a function on $\mathcal{P}^{-c}/\mathcal{O}$. In particular, $(\mathcal{F}\varphi^o)(\xi)=0$ if $\mathrm{ord}(\xi)<-c$.

Next we show that also $(\mathcal{F}\varphi^o)(\xi) = 0$ if $\mathrm{ord}(\xi) > -c$. The Fourier transform is

$$(\mathcal{F}\varphi^o)(\xi) = \int_{\mathcal{O}^\times} \chi^{-1}(t)\psi_\xi(t)\,\mathrm{d}t.$$

If $c = 1$ then $\mathrm{ord}(\xi) \geq 0$ and so ψ_ξ is trivial on $\mathcal{O}^\times$; but χ^{-1} is nontrivial on $\mathcal{O}^\times$, and so the previous display shows that $(\mathcal{F}\varphi^o)(\xi) = 0$. (Since χ^{-1} is a multiplicative character but the integral involves additive Haar measure, we can not merely quote the result that the integral of a nontrivial character over a compact group vanishes until we realize that the additive Haar measure is also multiplicative Haar measure on the unit group.) If $c > 1$ then we have a natural isomorphism between a multiplicative group of additive cosets and a multiplicative group of multiplicative cosets,

$$(\mathcal{O}/\mathcal{P}^{c-1})^\times \xrightarrow{\sim} \mathcal{O}^\times/(1+\mathcal{P}^{c-1}), \quad u + \mathcal{P}^{c-1} \longmapsto u(1+\mathcal{O}^{c-1}).$$

Consequently

$$\begin{aligned}(\mathcal{F}\varphi^o)(\xi) &= \sum_{\overline{u}\in(\mathcal{O}/\mathcal{P}^{c-1})^\times} \int_{\mathcal{P}^{c-1}} \chi^{-1}(u(1+t))\psi_\xi(u+t)\,\mathrm{d}t \\ &= \sum_{\overline{u}\in(\mathcal{O}/\mathcal{P}^{c-1})^\times} \chi^{-1}(u)\psi_\xi(u) \int_{\mathcal{P}^{c-1}} \chi^{-1}(1+t)\psi_\xi(t)\,\mathrm{d}t.\end{aligned}$$

But ψ_ξ is trivial on $\mathcal{P}^{c-1}$ and χ^{-1} is nontrivial on $1+\mathcal{P}^{c-1}$, and so the integral is 0, again because we are integrating over a compact subgroup of the unit group, so that the additive Haar measure is also multiplicative.

The remaining case is $\xi = u\varpi^{-c}$ where $u \in \mathcal{O}^\times$. The Fourier transform is

$$\begin{aligned}(\mathcal{F}\varphi^o)(\xi) &= \int_{\mathcal{O}^\times} \chi^{-1}(t)\psi_{u\varpi^{-c}}(t)\,\mathrm{d}t \\ &= \chi(u)\int_{\mathcal{O}^\times} \chi^{-1}(ut)\psi_{\varpi^{-c}}(ut)\,\mathrm{d}(ut) \\ &= \chi(\varpi^c\xi)\int_{\mathcal{O}^\times} \chi^{-1}(t)\psi(\varpi^{-c}t)\,\mathrm{d}t.\end{aligned}$$

Rewrite the previous display as

$$(\mathcal{F}\varphi^o)(\xi) = \chi(\varpi^c\xi)q^{-c/2}\tau(\chi,\psi),$$

where $\tau(\chi,\psi)$ is the normalized Gauss sum,

$$\tau(\chi,\psi) = q^{c/2}\int_{\mathcal{O}^\times} \chi^{-1}(t)\psi(\varpi^{-c}t)\,\mathrm{d}t.$$

(To see that the normalization is correct, compute that the square of the modulus of the integral is

$$\int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times}\chi^{-1}(t)\overline{\chi^{-1}}(t')\psi(\varpi^{-c}(t-t'))\,\mathrm{d}t'\,\mathrm{d}t$$
$$=\int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times}\chi^{-1}(t)\overline{\chi^{-1}}(tt')\psi(\varpi^{-c}(1-t')t)\,\mathrm{d}t'\,\mathrm{d}t$$
$$=\int_{\mathcal{O}^\times}\overline{\chi^{-1}}(t')\int_{\mathcal{O}^\times}\psi(\varpi^{-c}(1-t')t)\,\mathrm{d}t\,\mathrm{d}t'$$
$$=\int_{\mathcal{O}^\times}\overline{\chi^{-1}}(t')\int_{\mathcal{O}}\psi(\varpi^{-c}(1-t')t)\,\mathrm{d}t$$

dt'

$$-\int_{\mathcal{O}^\times}\overline{\chi^{-1}}(t')\int_{\mathcal{P}}\psi(\varpi^{-c}(1-t')t)\,\mathrm{d}t\,\mathrm{d}t'.$$

The first inner integral vanishes unless $1-t'\in\mathcal{P}^c$, in which case it is $\mu(\mathcal{O})=1$, and then the outer integral is $\mu(1+\mathcal{P}^c)=\mu(\mathcal{P}^c)=q^{-c}$. So overall the first double integral is q^{-c}. The second double integral is similar except that now the inner integral vanishes unless $1-t'\in\mathcal{P}^{c-1}$, in which case the outer integral vanishes. Thus the modulus of the integral is $q^{-c/2}$ as desired.)

Now we have

$$(\mathcal{F}\varphi^o)(\xi)=q^{-c/2}\tau(\chi,\psi)\overline{\varphi^o}(\varpi^c\xi).$$

And $\overline{\varphi^o}$ is the basic function for χ^{-1}. Consequently the formula (3) for ε gives

$$\begin{aligned}\varepsilon(s,\chi,\psi)&=q^{-c/2}\tau(\chi,\psi)\langle\mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,R_{\varpi^c}\overline{\varphi^o}\rangle\\&=q^{-c/2}\tau(\chi,\psi)\langle R_{\varpi^{-c}}\mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\overline{\varphi^o}\rangle\\&=q^{-c/2}\tau(\chi,\psi)\chi^{-1}(\varpi^{-c})|\varpi^{-c}|^{1-s}\langle\mu_{\chi^{-1}|\cdot|^{1-s}}\,,\,\overline{\varphi^o}\rangle\\&=q^{-c/2}\tau(\chi,\psi)\chi(\varpi^c)q^{(1-s)c}\\&=q^{(1/2-s)c}\tau(\chi,\psi)\chi(\varpi^c).\end{aligned}$$

So far the calculation has assumed that ψ has conductor 0. The general case is $\psi=\psi^o_{\varpi^\nu}$ where ψ^o has conductor 0. By the second reduction identity,

$$\begin{aligned}\varepsilon(s,\chi,\psi)&=q^{(1/2-s)\nu}\chi(\varpi^\nu)q^{(1/2-s)c}\tau(\chi,\psi^o)\chi(\varpi^c)\\&=q^{(1/2-s)(c+\nu)}\chi(\varpi^{c+\nu})\tau(\chi,\psi^o).\end{aligned}$$

To express the Gauss sum in terms of ψ rather than ψ^o, compute

$$\begin{aligned}\tau(\chi,\psi^o)&=q^{c/2}\int_{\mathcal{O}^\times}\chi^{-1}(t)\psi^o(\varpi^{-c}t)\,\mathrm{d}^ot\\&=q^{1/2(c+\nu)}\int_{\mathcal{O}^\times}\chi^{-1}(t)\psi_{\varpi^{-\nu}}(\varpi^{-c}t)\,\mathrm{d}t\\&=q^{1/2(c+\nu)}\int_{\mathcal{O}^\times}\chi^{-1}(t)\psi(\varpi^{-c-\nu}t)\,\mathrm{d}t,\end{aligned}$$

and so the more general definition of the normalized Gauss sum is

$$\tau(\chi,\psi)=q^{1/2(c+\nu)}\int_{\mathcal{O}^\times}\chi^{-1}(t)\psi(\varpi^{-c-\nu}t)\,\mathrm{d}t.$$

In sum,

$$\boxed{\varepsilon(s,\chi,\psi) = q^{(1/2-s)(c+\nu)}\tau(\chi,\psi)\chi(\varpi^{c+\nu}), \quad F = \text{na}, \ \chi \text{ ramified.}}$$

9. The Global ε-Factor

Fix an archimedean place v. We have shown that the local ε-factor is

$$\varepsilon_v(s,\chi_v,\psi_v) = q_v^{(1/2-s)(c_v+\nu_v)}\tau_v(\chi_v,\psi_v)\chi_v(\varpi_v^{c_v+\nu_v}),$$

where

$$\begin{aligned} q_v &= |\mathcal{O}/\mathcal{P}|, \\ \varpi_v &= \text{a generator of } PF_v, \\ c_v &= \text{the conductor of } \chi_v, \\ \nu_v &= \text{the conductor of } \psi_v, \end{aligned}$$

$$\tau_v(\chi_v,\psi_v) = \begin{cases} 1 & \chi_v \text{ unramified,} \\ q_v^{1/2(c_v+\nu_v)} q^{1/2(c+\nu)} \int_{\mathcal{O}_v^\times} \chi_v^{-1}(t)\psi_v(\varpi_v^{-c_v-\nu_v}t)\,\mathrm{d}t & \chi_v \text{ ramified.} \end{cases}$$

Replace ψ by ψ_a where $a \in \mathbf{k}^\times$. At each nonarchimedean place we have from before,

$$\varepsilon_v(s,\chi_v,\psi_{v,a}) = |a|_v^{s-1/2}\chi_v(a_v)\varepsilon_v(s,\chi_v,\psi_v).$$

Assuming that the same formula holds for archimedean places, the global product of the local ε-factors is independent of ψ, and so it is written

$$\varepsilon(s,\chi) = \prod_v \varepsilon_v(s,\chi_v,\psi_v).$$

Part 4. The Final Calculation

To set up the final calculation, define

$$\begin{aligned} \Lambda(s,\chi) &= \prod_v L_v(s,\chi_v), \\ Z_o(s,\chi) &= \prod_v Z_{o,v}(s,\chi_v), \\ Z(s,\chi) &= \prod_v Z_v(s,\chi_v). \end{aligned}$$

Then

$$\begin{aligned} \Lambda(s,\chi)Z_o(s,\chi) &= Z(s,\chi) && \text{multiplying together each } Z_v \\ &= (\mathcal{F}Z)(1-s,\chi^{-1}) && \text{by the global functional equation} \\ &= \prod_v (\mathcal{F}Z_v)(1-s,\chi_v^{-1}) && \text{decomposing } \mathcal{F}Z \\ &= \prod_v L_v(1-s,\chi_v^{-1})(\mathcal{F}Z_{o,v})(1-s,\chi_v^{-1}) && \text{factoring each } \mathcal{F}Z_v \\ &= \prod_v L_v(1-s,\chi_v^{-1})\varepsilon_v(s,\chi_v,\psi_v)Z_{o,v}(s,\chi_v) && \text{by the local functional equations} \\ &= \Lambda(1-s,\chi^{-1})\varepsilon(s,\chi)Z_o(s,\chi) && \text{assembling } \Lambda,\ \varepsilon, \text{ and } Z_o. \end{aligned}$$

Therefore

$$\boxed{\Lambda(s,\chi) = \varepsilon(s,\chi)\Lambda(1-s,\chi^{-1}).}$$

Part 5. Application: Quadratic ε-Factors are Trivial

Let $\mathbf{k}$ be a quadratic number field, and let χ be its character. Note that $\chi^{-1} = \chi$. Compute that

$$\begin{aligned} Z_{\mathbf{k}}(1-s) &= Z_{\mathbf{k}}(s) \\ &= Z_{\mathbb{Q}}(s)\Lambda_{\mathbb{Q}}(s,\chi) \\ &= Z_{\mathbb{Q}}(1-s)\varepsilon(s,\chi)\Lambda_{\mathbb{Q}}(1-s,\chi) \\ &= \varepsilon(s,\chi)Z_{\mathbf{k}}(1-s). \end{aligned}$$

Thus $\varepsilon(s,\chi) = 1$. This result encompasses the value of the quadratic Gauss sum.