

## FINITELY-GENERATED ABELIAN GROUPS

**Structure Theorem for Finitely-Generated Abelian Groups.** *Let  $G$  be a finitely-generated abelian group. Then there exist*

- a nonnegative integer  $t$  and integers  $1 < d_1 \mid d_2 \mid \cdots \mid d_t$ ,
- a nonnegative integer  $r$

such that  $G$  takes the form

$$G \cong \mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z} \times \mathbf{Z}^r.$$

The integers  $1 < d_1 \mid d_2 \mid \cdots \mid d_t$  are called the **elementary divisors** of  $G$ , and the subgroup

$$G_{\text{tor}} = \mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z}$$

is called the **torsion subgroup** of  $G$ . The nonnegative integer  $r$  is called the **rank** of  $G$ , and the subgroup

$$G_{\text{free}} = \mathbf{Z}^r$$

is called the **free subgroup** of  $G$ . The elementary divisors of  $G$  and the rank of  $G$  are uniquely determined by  $G$ , but the summands of the torsion subgroup and of the free subgroup need not be. The case  $t = r = 0$  is understood to mean that the group is trivial.

**Proof.** The group  $G$  is described by a set of  $r$  integer-linear relations on  $g$  generators,

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1g}x_g = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2g}x_g = 0 \\ \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rg}x_g = 0 \end{array} \right\}.$$

More concisely, the relations are

$$\sum_{j=1}^g a_{ij}x_j = 0, \quad i = 1, \dots, r.$$

Even more concisely, the relations encode as an  $r \times g$  integer matrix,

$$A = [a_{ij}]_{r \times g}.$$

However, *the relations (and the matrix) are not uniquely determined by the group.* The following operations on the relations preserve the group that the data describe.

- *Relation recombine.* Replace the  $i$ th relation by itself plus  $k$  times the  $j$ th relation. Here  $i, j \in \{1, \dots, r\}$  with  $j \neq i$ , and  $k \in \mathbf{Z}$ . In symbols,  $r_i \leftarrow r_i + kr_j$ .
- *Relation scale.* Negate the  $i$ th relation. Here  $i \in \{1, \dots, r\}$ . In symbols,  $r_i \leftarrow -r_i$ .
- *Relation transposition.* Exchange the  $i$ th and the  $j$ th relations. Here again  $i, j \in \{1, \dots, r\}$  with  $j \neq i$ . In symbols,  $r_i \leftrightarrow r_j$ .

Also, the following operations on the generators preserve the group that the data describe.

- *Generator recombine.* Replace the  $j$ th generator by itself minus  $k$  times the  $i$ th generator. Here  $i, j \in \{1, \dots, g\}$  with  $i \neq j$ , and  $k \in \mathbf{Z}$ . In symbols,  $x_j \leftarrow x_j - kx_i$ . (This operation is described slightly differently from the relation recombine above in that  $i$  and  $j$  have exchanged roles and  $k$  is negated. The reason for modifying the description will manifest itself in a common description of the two recombiners, to arise in a moment.)
- *Generator scale.* Negate the  $j$ th generator. Here  $i \in \{1, \dots, g\}$ . In symbols,  $x_i \leftarrow -x_i$ .
- *Generator transposition.* Exchange the  $i$ th and the  $j$ th generators. Here again  $i, j \in \{1, \dots, g\}$  with  $j \neq i$ . In symbols,  $x_i \leftrightarrow x_j$ .

The various operations on the data for  $G$  translate into row operations and column operations on the describing matrix  $A$  for  $G$  as follows (Here  $r$  stands for *row* and  $c$  for *column*.)

- *Recombine.*  $r_i \leftarrow r_i + kr_j$  or  $c_i \leftarrow c_i + kc_j$ .
- *Scale.*  $r_i \leftarrow -r_i$  or  $c_i \leftarrow -c_i$ .
- *Transposition.*  $r_i \leftrightarrow r_j$  or  $c_i \leftrightarrow c_j$ .

The recombine operation here is the common description of the two recombine operations above. The operations here are similar to the recombine, scale, and transposition operations that arise in solving a system of linear equations, but the analogy is imperfect. In our context, the matrix  $A$  represents the data describing a finitely-generated abelian group, and its entries are integers. Here we are allowed row operations and column operations, but we may scale only by  $-1$ . (Actually, we may scale vacuously by  $1$  as well. The real point is that we may scale rows or columns by any invertible integer, i.e., by  $\pm 1$ ; whereas in linear algebra we could scale rows by any invertible field element, i.e., by any nonzero field element.)

Now to establish the structure of a given finitely-generated abelian group with describing matrix  $A$ , proceed as follows.

On the one hand, if  $A$  is the zero matrix then there are  $g$  generators free of any constraining relations,

$$G \cong \mathbf{Z}^g.$$

That is,  $t = 0$  and there are no elementary divisors, and  $r = g$ , and we are done.

On the other hand, if  $A$  is not the zero matrix then carry out row and column operation to make the upper left entry of  $A$  as small as possible a positive integer  $b_1$  that can be placed there in finitely many steps,

$$A \leftarrow \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rg} \end{bmatrix}.$$

(So here the entries  $a_{ij}$  need not be the original  $a_{ij}$ . The  $a_{ij}$  will continue to vary throughout the calculation as it proceeds.) Then in fact  $b_1 \mid a_{1j}$  for  $j = 2, \dots, g$  (else we could make a smaller positive upper left entry), so that after further column operations we may take  $a_{1j} = 0$  for  $j = 2, \dots, g$ . Similarly we may take  $a_{i1} = 0$  for  $i = 2, \dots, r$ . And now the same ideas show that  $b_1 \mid a_{ij}$  for  $i = 2, \dots, g$  and

$j = 2, \dots, r$ . That is, in fact

$$A \leftarrow \left[ \begin{array}{c|ccc} b_1 & 0 & \cdots & 0 \\ \hline 0 & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{r2} & \cdots & a_{rg} \end{array} \right], \quad 1 \leq b_1 \mid a_{ij} \text{ for all } i, j.$$

Repeating the process until it terminates, we eventually get

$$A \leftarrow \left[ \begin{array}{cccc|ccc} b_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_s & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right], \quad 1 \leq b_1 \mid b_2 \mid b_3 \mid \cdots \mid b_s.$$

Some of the diagonal entries  $b_i$  may equal 1. Eliminate their rows and columns, thus discarding pointless relations of the form  $g = 0$  and discarding spurious references to 0 in the relations among the generators. Also, eliminate the zero-rows since they merely stipulate the relation  $0 = 0$ . The result is a similar but possibly-smaller matrix with no 1's on the diagonal,

$$A \leftarrow \left[ \begin{array}{cccc|ccc} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_t & 0 & \cdots & 0 \end{array} \right], \quad 1 < d_1 \mid d_2 \mid \cdots \mid d_t.$$

If  $A$  is the null matrix then the group is trivial. Otherwise, the group is described by generators  $y_1, \dots, y_h$  (possibly  $h < g$  where  $g$  is the number of generators initially given), the first  $t$  of them subject to the relations

$$\begin{aligned} d_1 y_1 &= 0 \\ d_2 y_2 &= 0 \\ &\vdots \\ d_t y_t &= 0, \end{aligned}$$

and the remaining  $r = h - t$  generators free of relations. In other words, any element of  $G$  takes the form

$$z = c_1 y_1 + \cdots + c_t y_t + c_{t+1} y_{t+1} + \cdots + c_{t+r} y_{t+r}$$

where

$$0 \leq c_1 < d_1, \quad \dots, \quad 0 \leq c_t < d_t, \quad c_{t+j} \in \mathbf{Z} \text{ for } j = 1, \dots, r.$$

And thus as claimed,

$$G \cong \mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z} \times \cdots \times \mathbf{Z}/d_t\mathbf{Z} \times \mathbf{Z}^r.$$

The proof of uniqueness is omitted.