Mathematics 361: Number Theory
Assignment #3

Reading: Ireland and Rosen, Chapter 3 (including the exercises)

Problems:

1. Consider the arithmetic functions

$$\mathcal{A} = \{f : \mathbf{Z}^+ \longrightarrow \mathbf{C}, f(1) \neq 0\},$$

consider the convolution

$$* : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}, \qquad (f * g)(n) = \sum_{d|n} f(d)g(n/d),$$

and consider also the particular arithmetic function $\mathbf{i} \in \mathcal{A}$ given by

$$\mathbf{i}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $(\mathcal{A}, *)$ forms an abelian group by showing that $*$ is commutative and associative, that $f * \mathbf{i} = f$ for all $f \in \mathcal{A}$ and that the recursive definition

$$f^{-1}(n) = \begin{cases} 1/f(1) & \text{if } n = 1, \\ -1/f(1) \cdot \sum_{\substack{d|n \\ d<n}} f^{-1}(d)f(n/d) & \text{if } n > 1 \end{cases}$$

makes sense and indeed produces an inverse of f.

2. Prove yet again that there are infinitely many primes by working Ireland and Rosen, Exercise 2.3 or Exercises 2.4–2.5 or Exercises 2.6–2.8. (Read them all. See also Chapter 1 of Ribenboim's **Book of Prime Number Records** if you want more.)

3. Get some practice with arithmetic functions by working a selection from Ireland and Rosen, Exercises 2.9–2.21.

4. Acquire some initial familiarity with the Riemann zeta function of analytic number theory by reading and perhaps working Ireland and Rosen, Exercises 2.25–2.27.